U N I V E R S I T Y   O F   B E R G E N

Algorithms Research Group

On Sparsification for Computing Treewidth

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Outline

Treewidth

Sparsification

Results

- Sparsification lower bound for TREEWIDTH
- Quadratic-vertex kernel upper bound for TREEWIDTH [VC]

Conclusion

Treewidth

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  - NP-complete [Arnborg et al.’87]

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- Possible ways to make G' provably simpler than G:
  - Upper bound on the density of G'
  - Upper bound on the vertex count of G', in terms of structural measures of G

Parameterized Complexity and Kernelization

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- It is a kernelization (or **kernel**) if $\mathbb{Q} = \mathbb{Q}'$

Sparsification Analysis using Kernelization

- Based on the parameterization by vertex count:
  - *n*-TREEWIDTH

    Input: $n \in \mathbb{N}$, an n-vertex graph G, an integer k

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Theorem. *n*-TREEWIDTH does not have a generalized kernel of bitsize $\mathcal{O}(n^{2-\varepsilon})$, for any $\varepsilon > 0$, unless NP ⊆ coNP/poly

Proof Technique

- Proof using **cross-composition of bounded cost**
  - Introduced in the journal version of the paper on cross-composition [Bodlaender, J, Kratsch ’12]
  - Easier front-end to the complementary witness lemma of Dell & van Melkebeek [STOC’10]

Proof Technique

Corollary [Bodlaender et al.'12].
If there is a polynomial-time algorithm that:
- composes the OR of t^2 similar size-s instances of an NP-hard problem,
- into an instance (G^*, n^*, k^*) of *n*-TREEWIDTH with $n^* \in \mathcal{O}(t \cdot s^{\mathcal{O}(1)})$,

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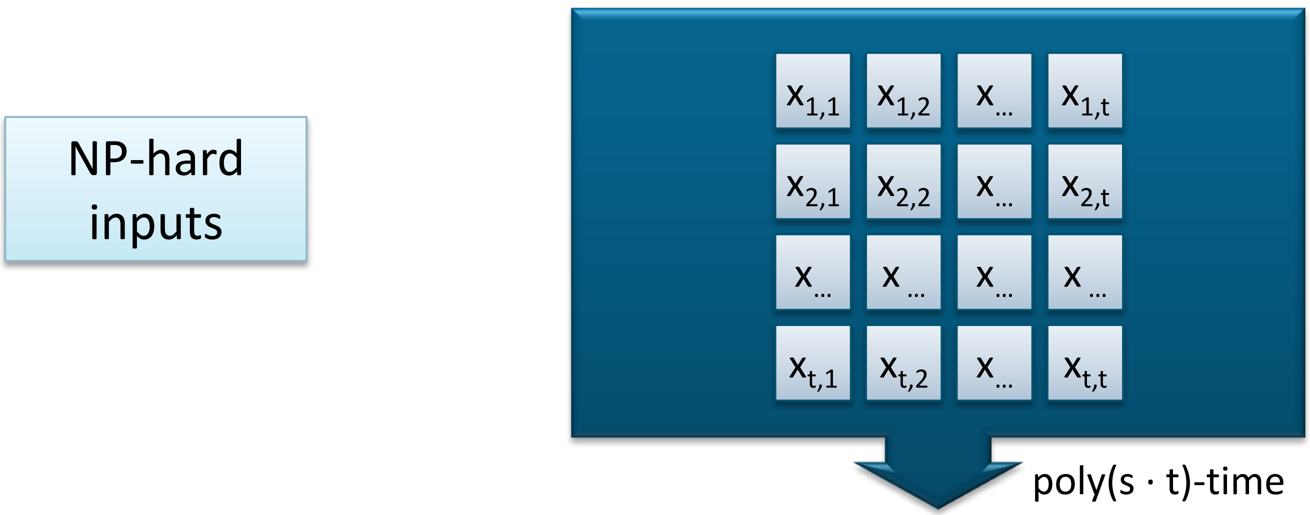

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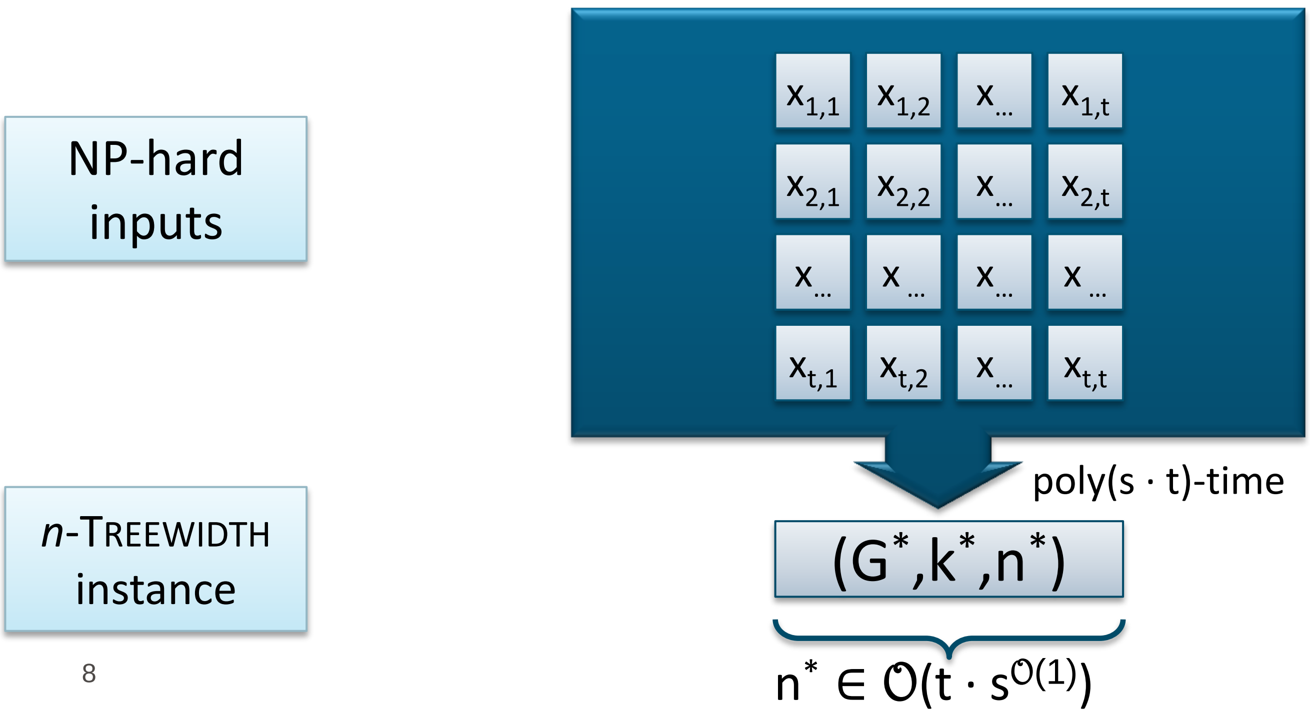

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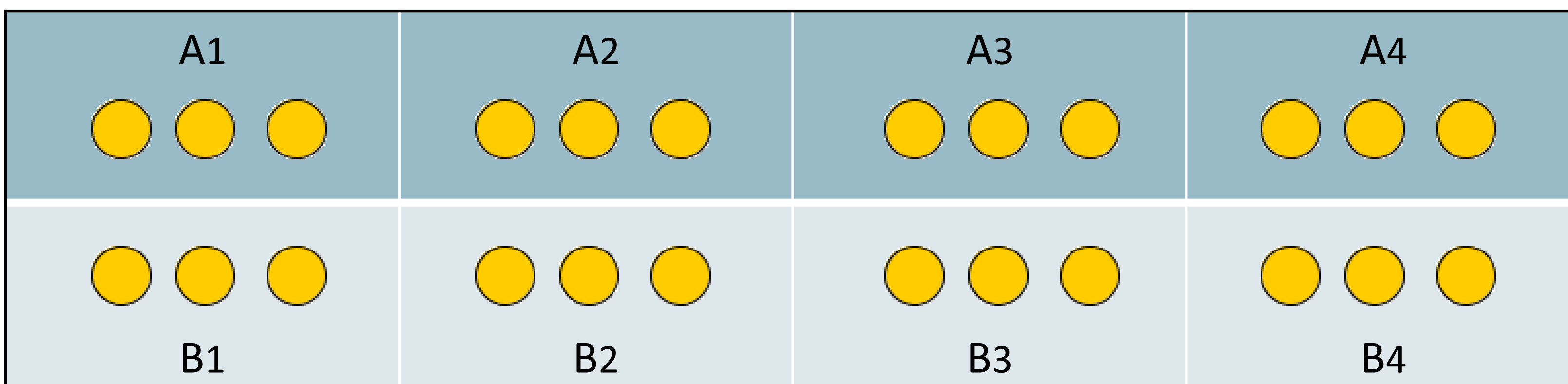

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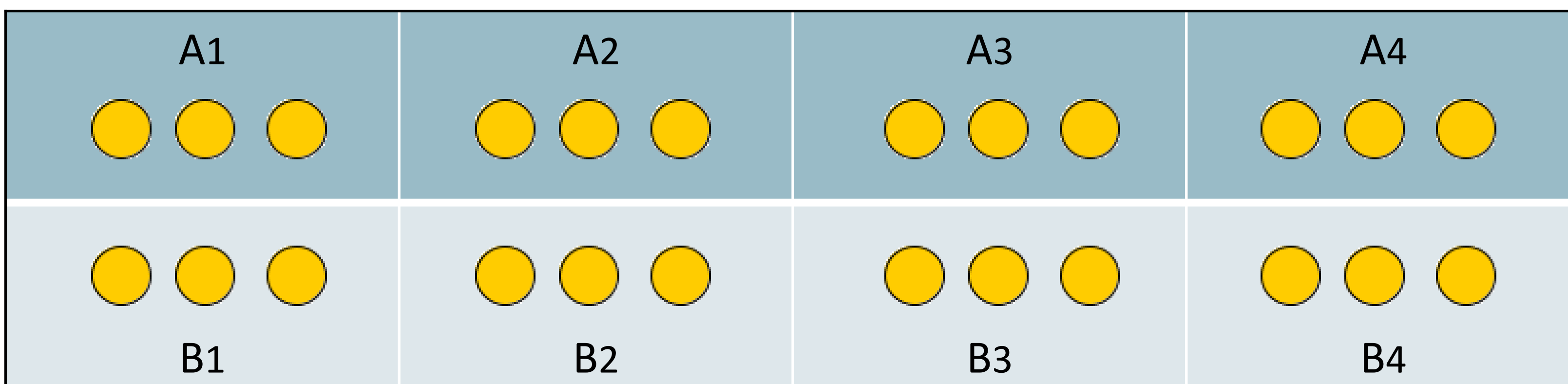

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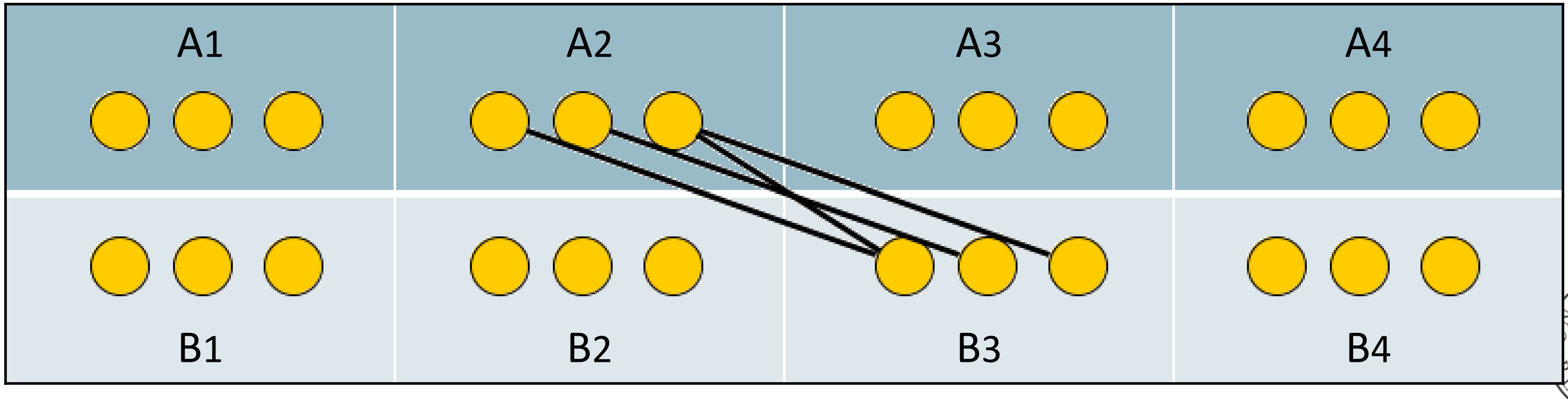

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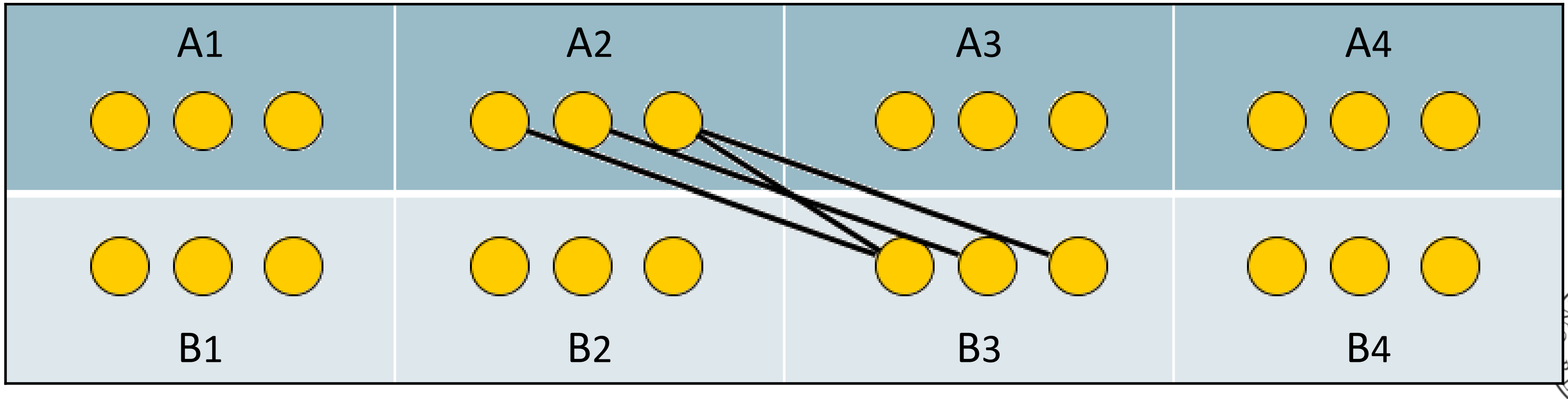

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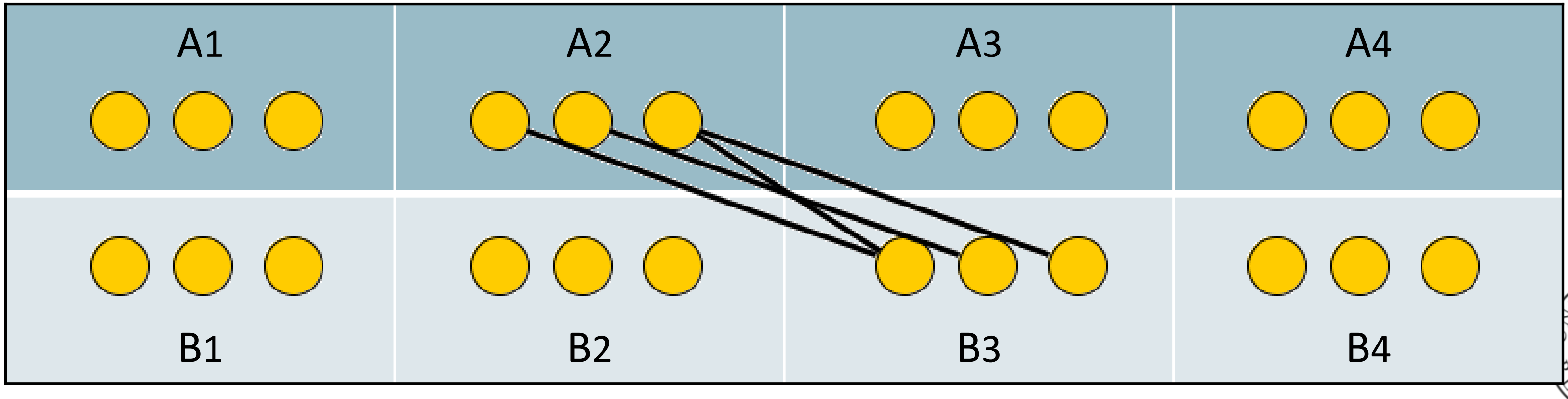

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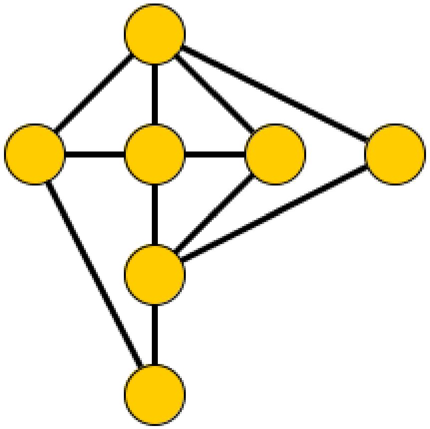

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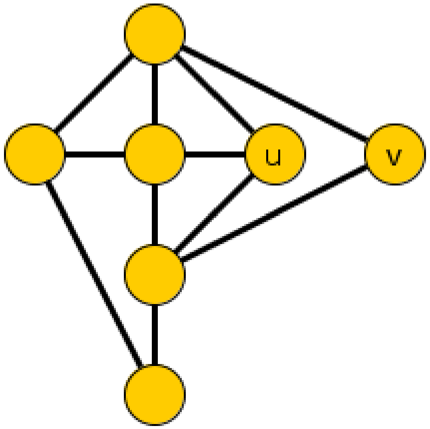

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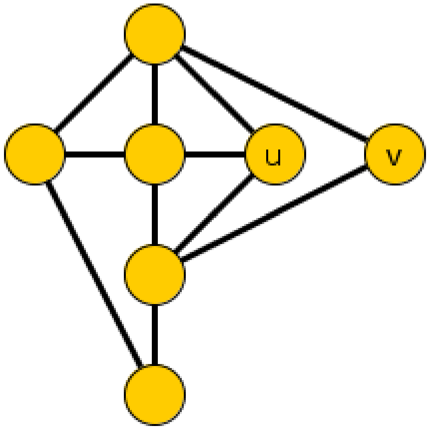

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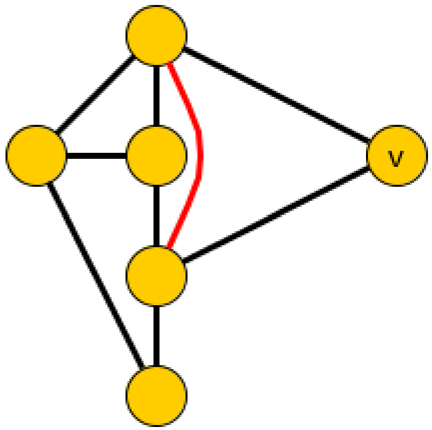

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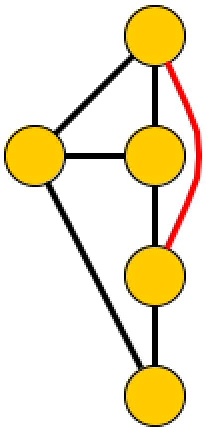

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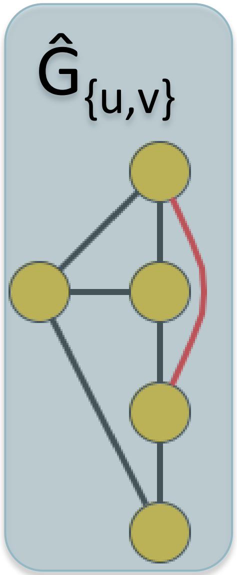

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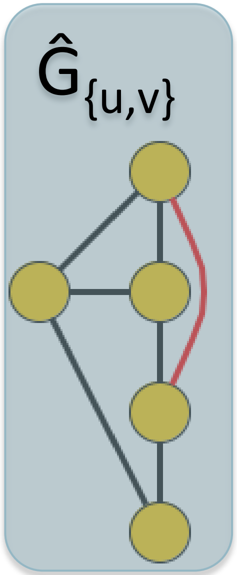

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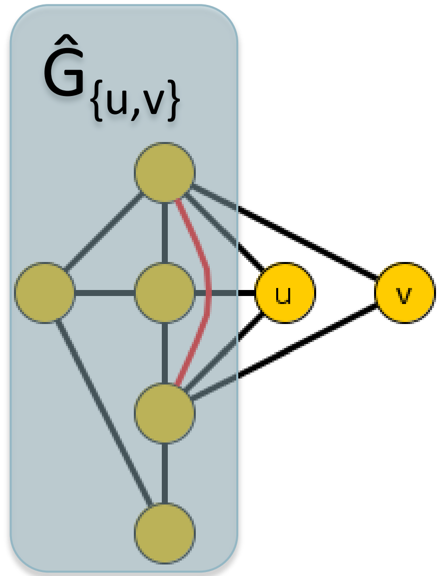

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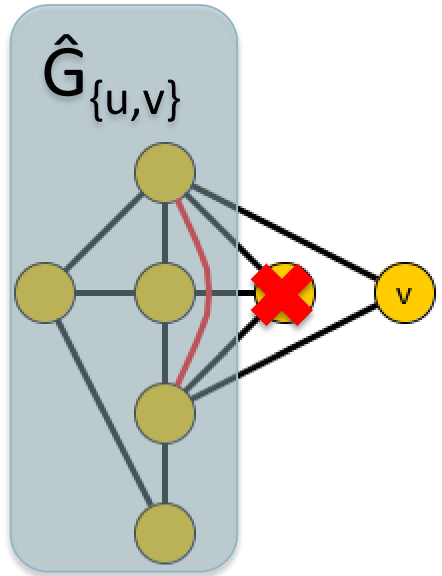

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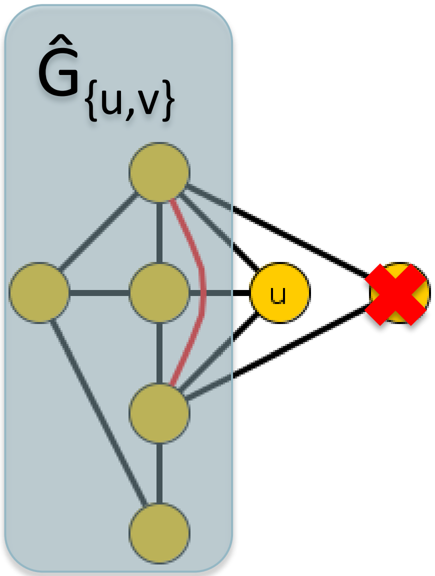

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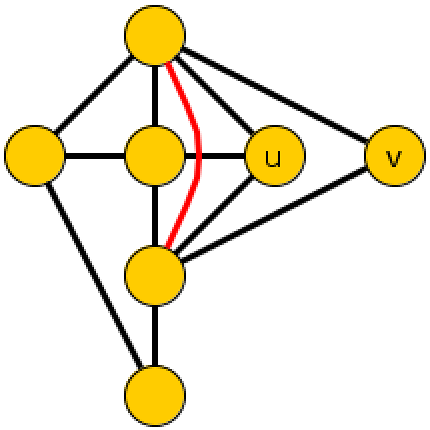

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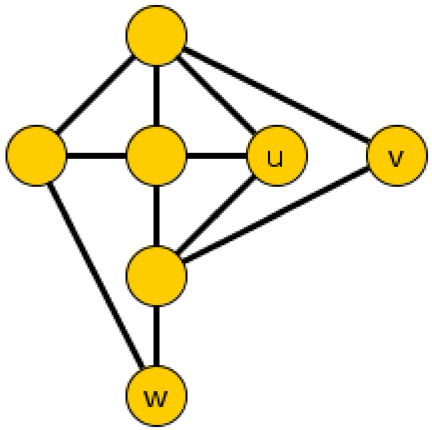

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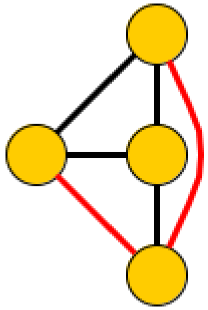

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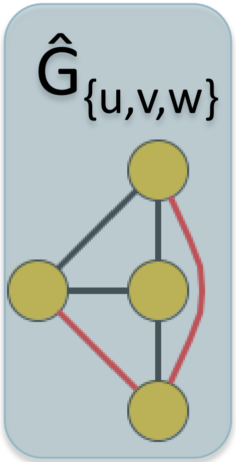

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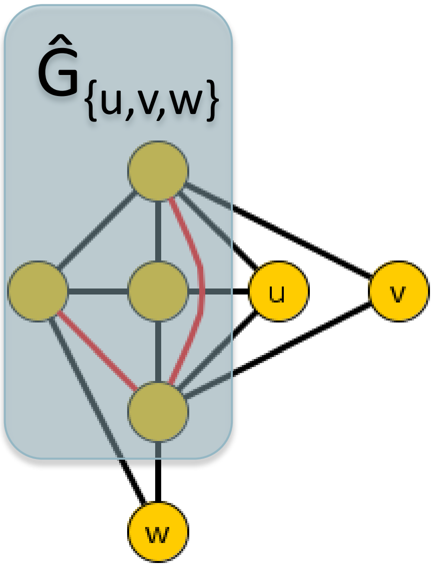

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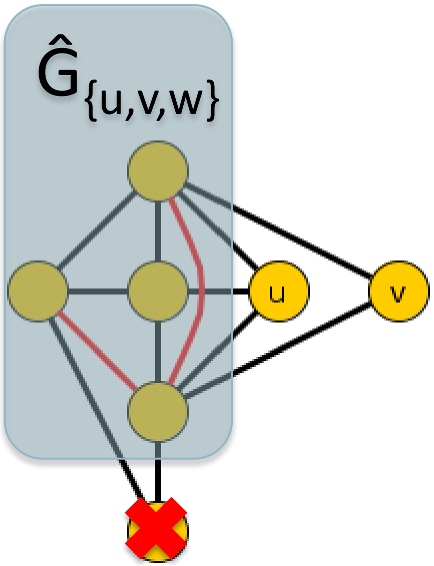

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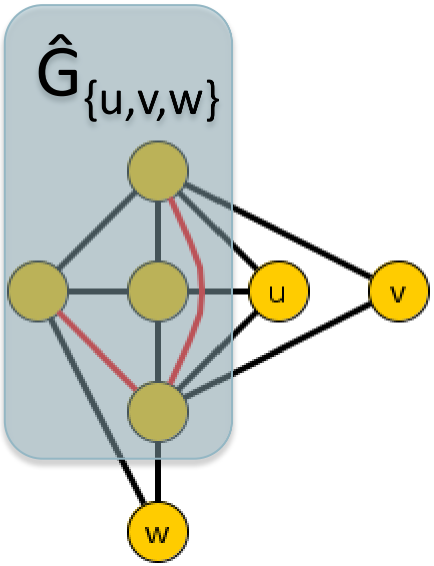

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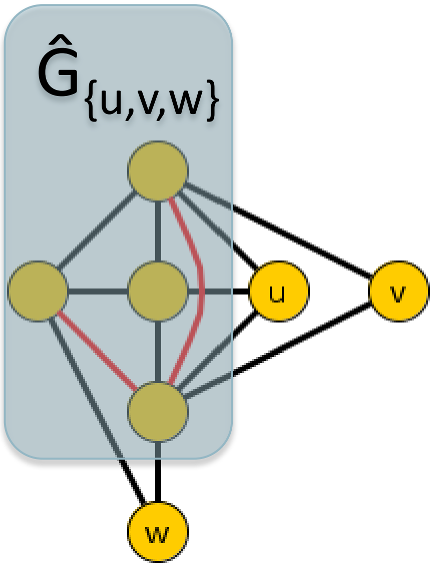

Lemma. If T is a treewidth-invariant set in G, then $\textsc{tw}(G) = \max\{\textsc{tw}(\hat{G}_T), \Delta(T)\}$

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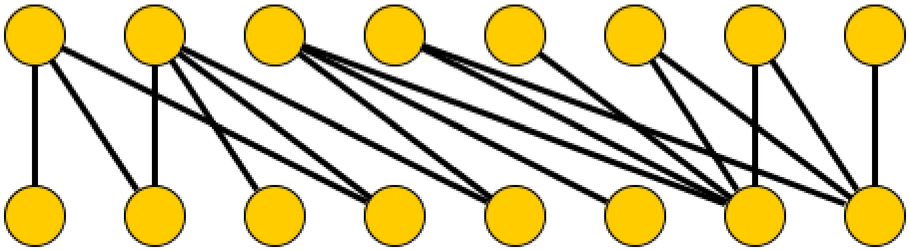

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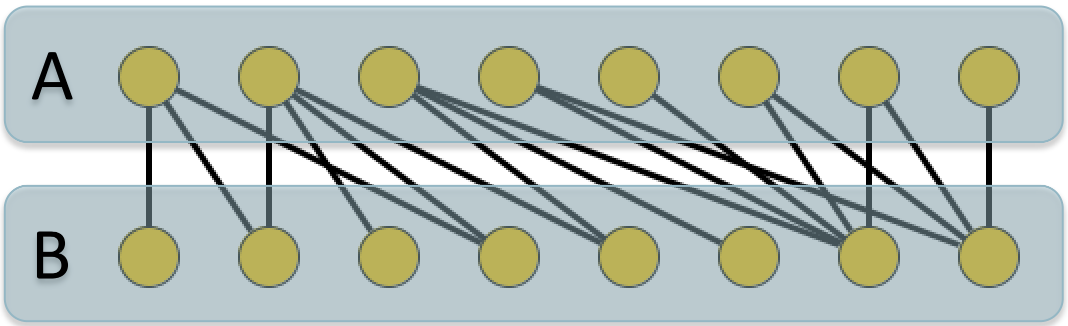

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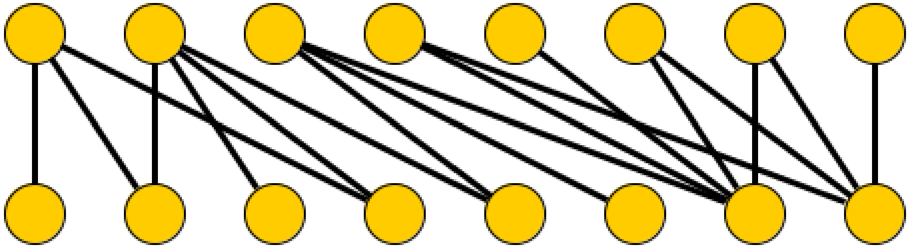

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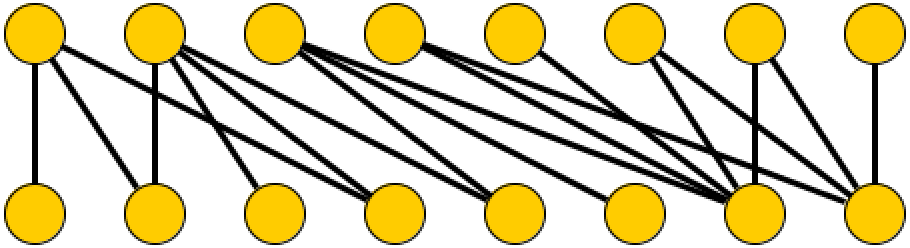

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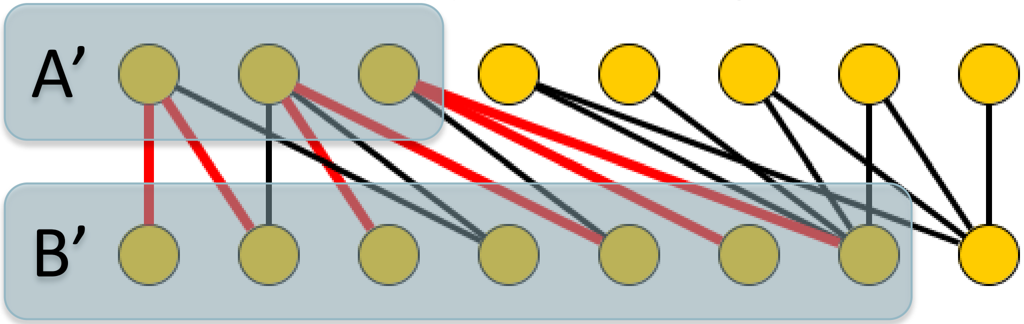

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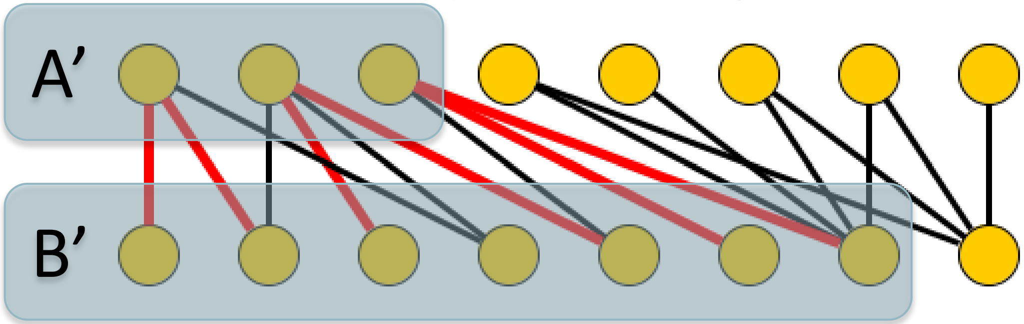

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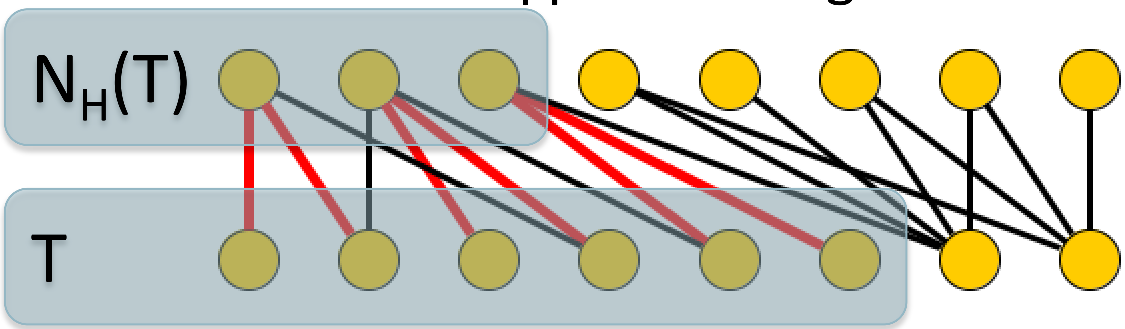

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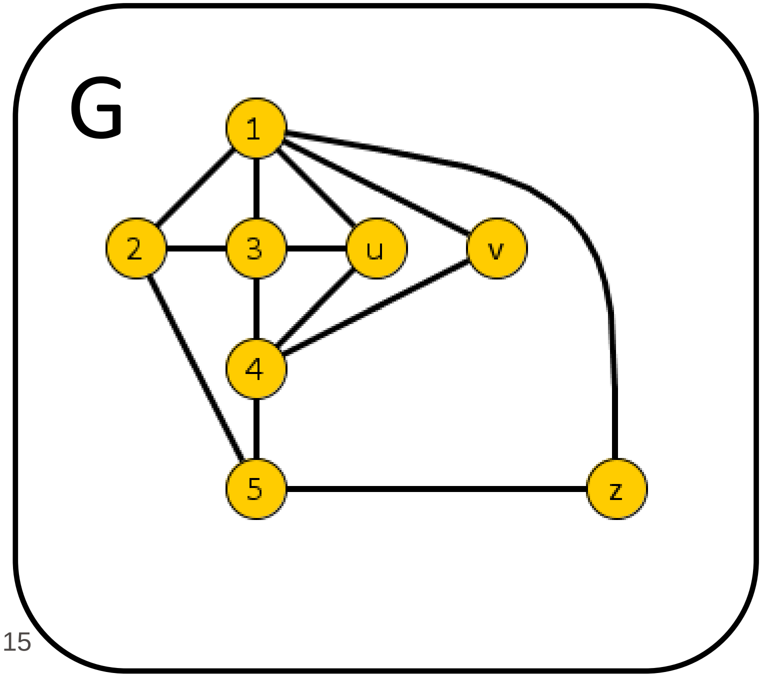

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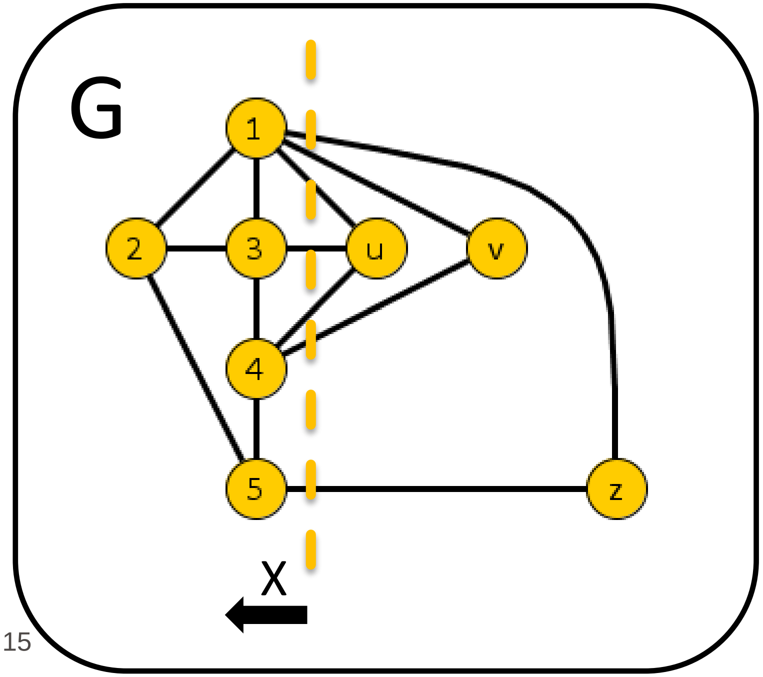

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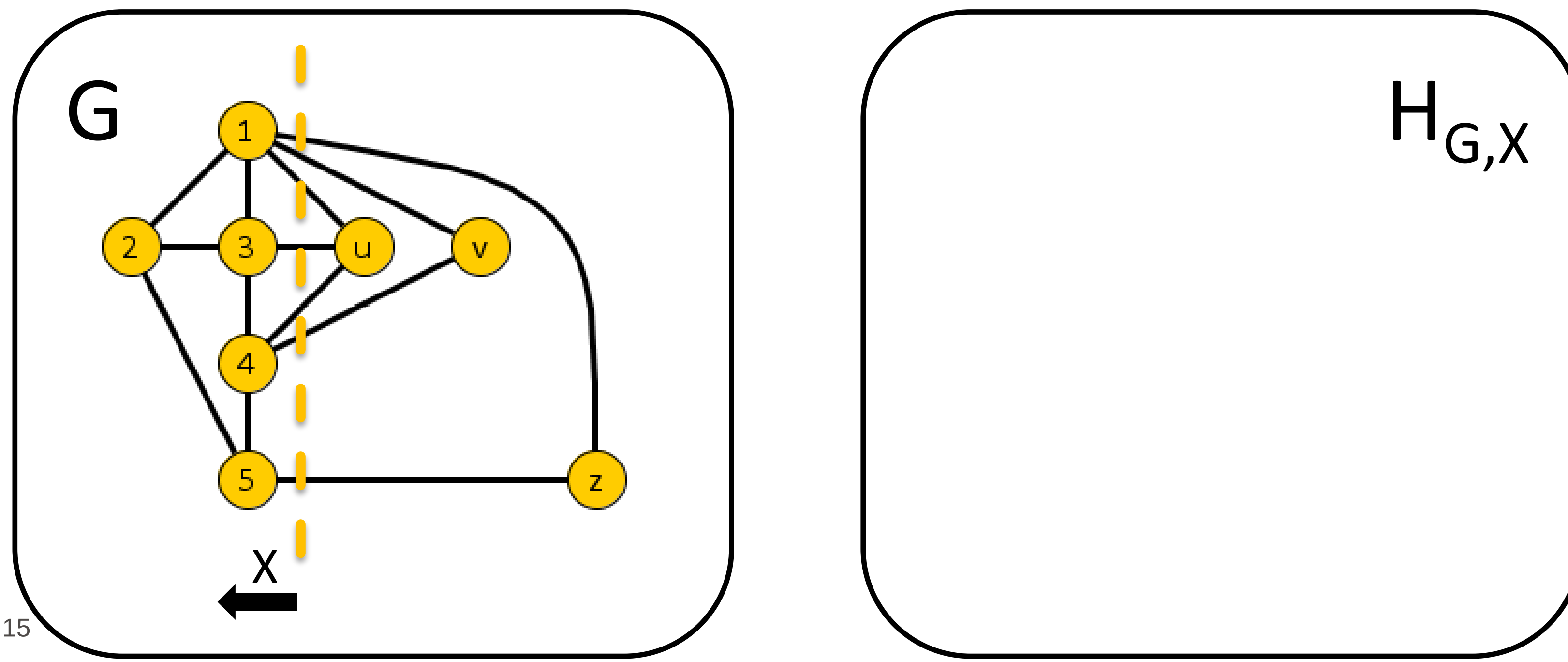

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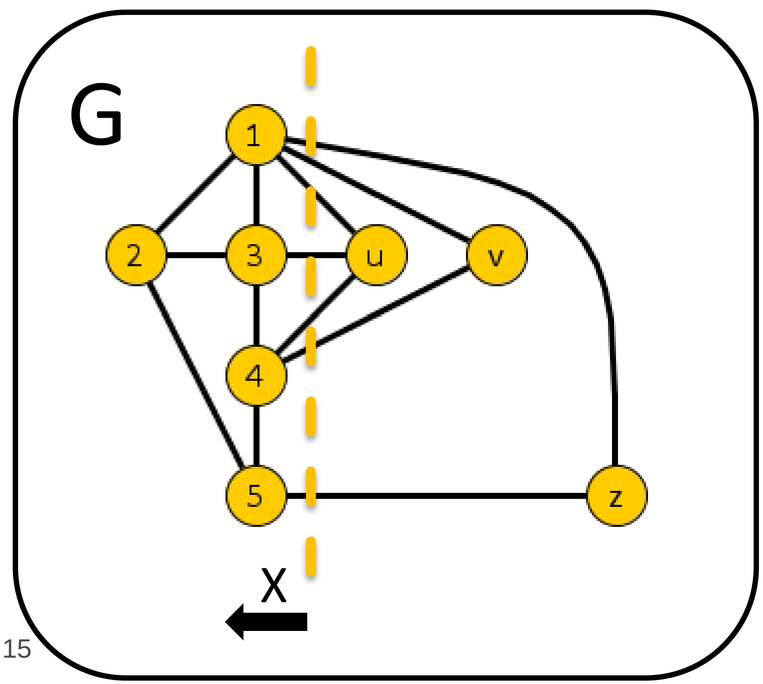

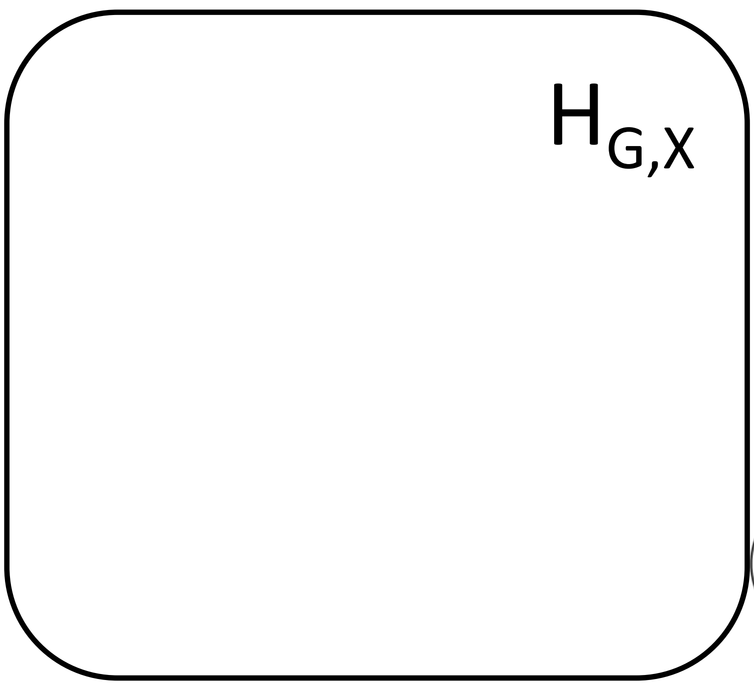

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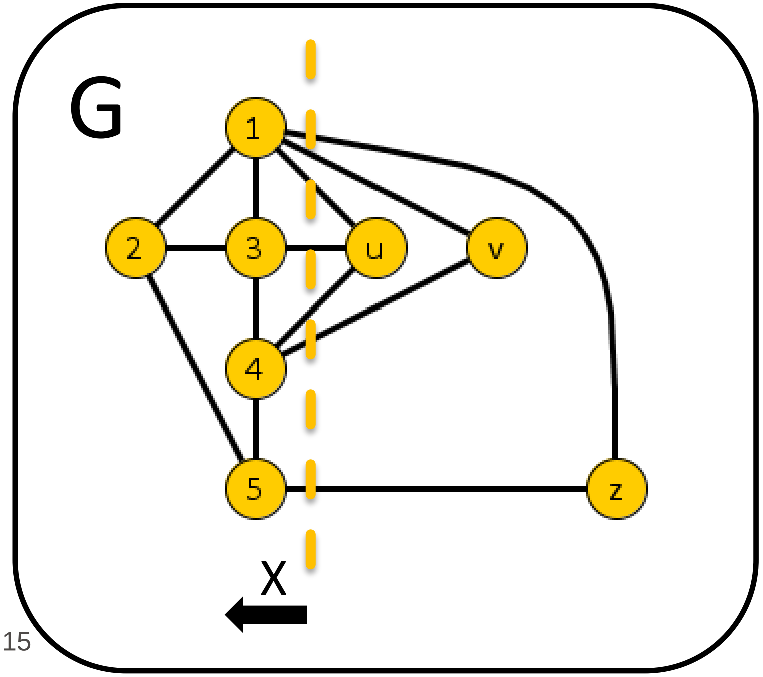

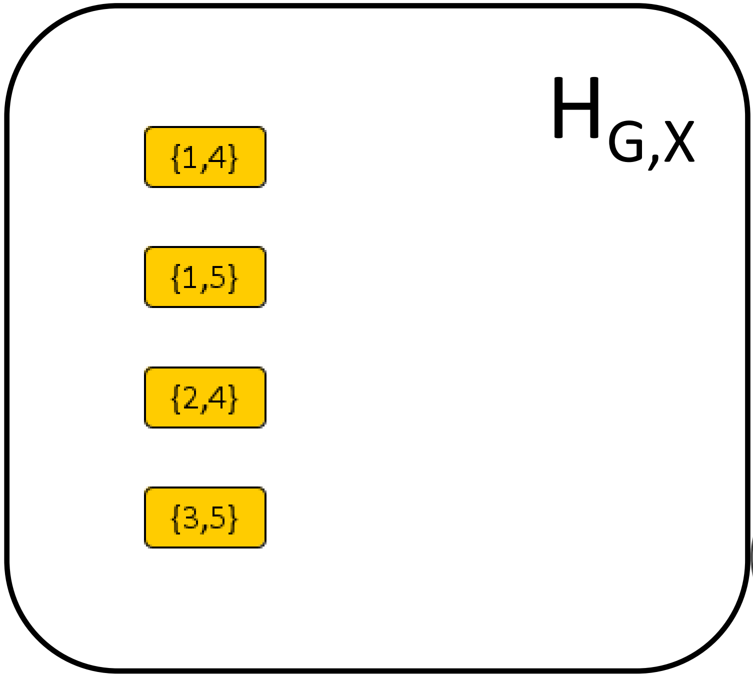

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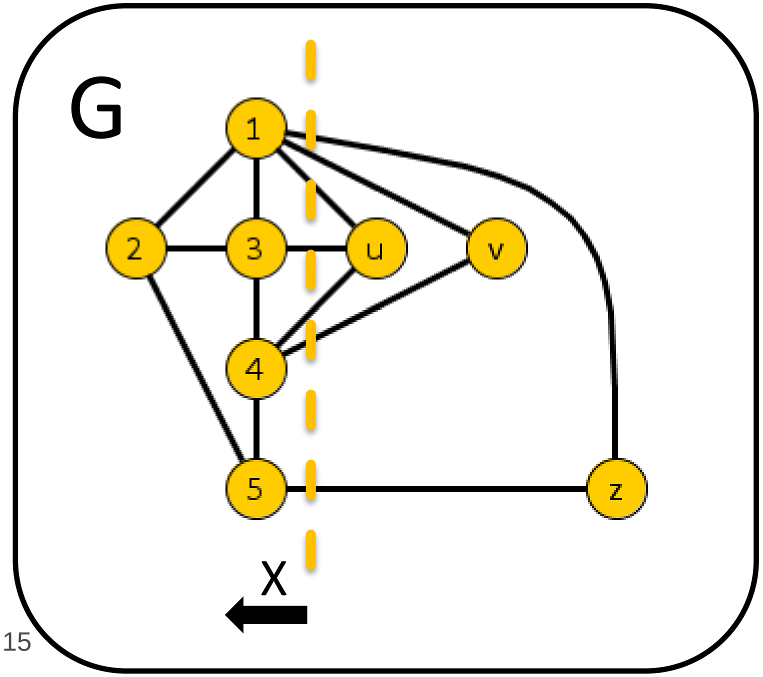

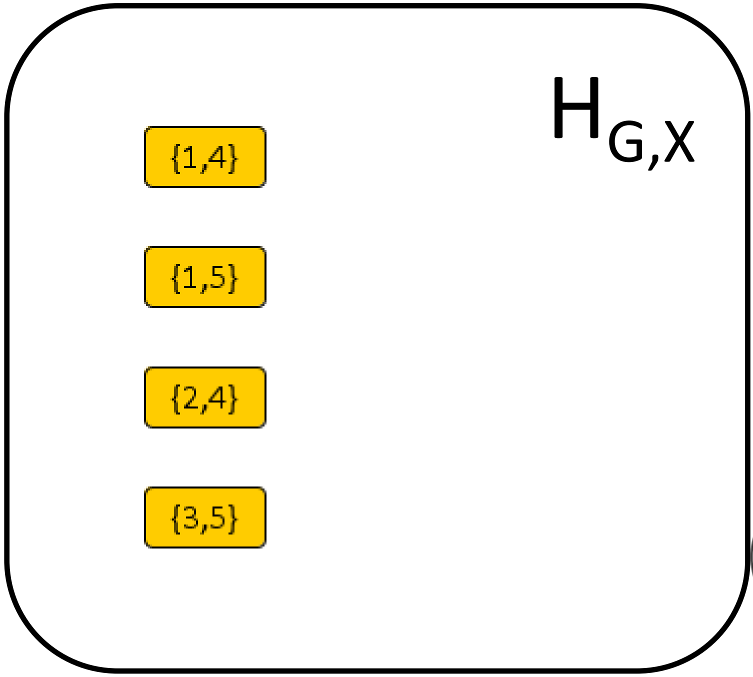

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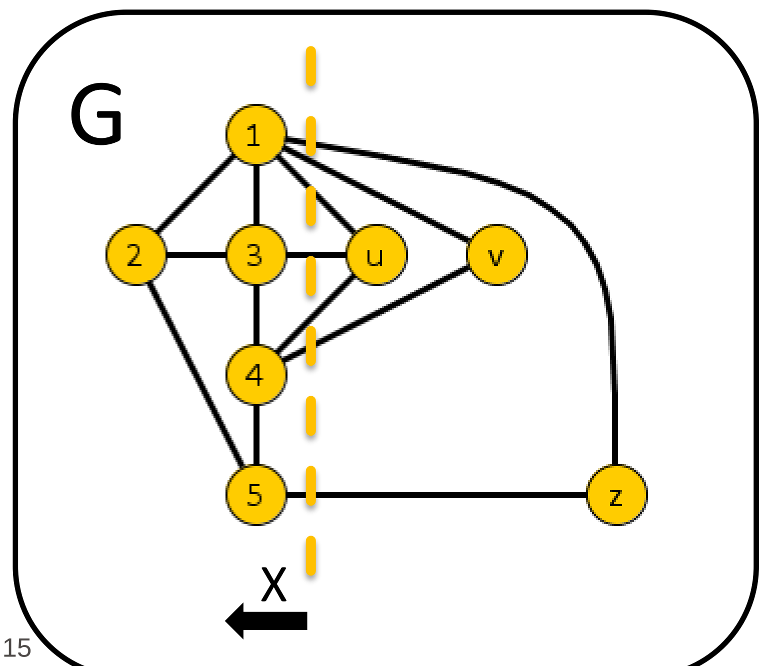

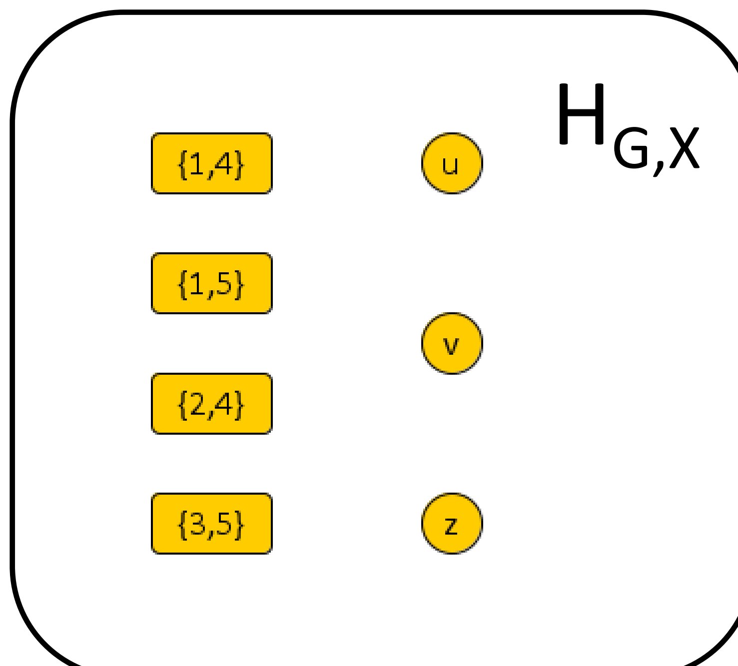

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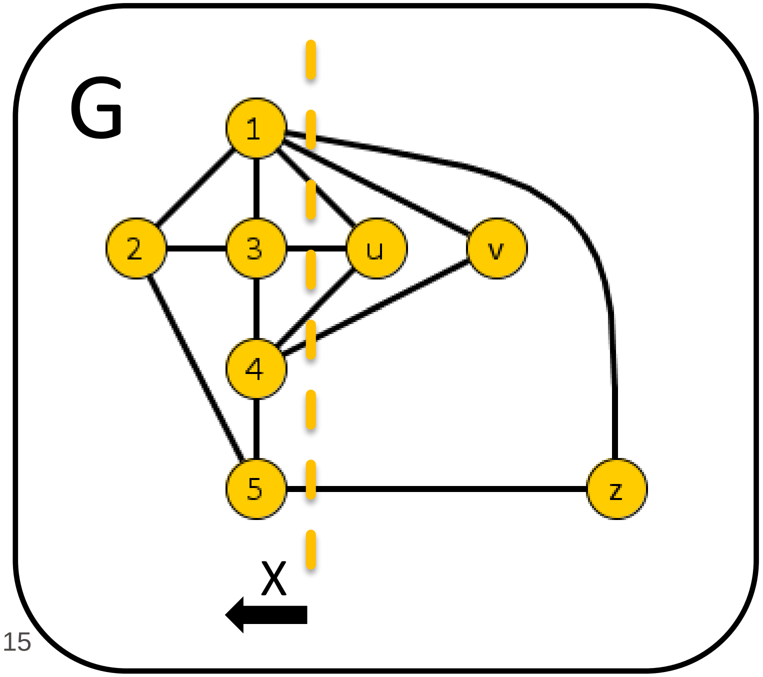

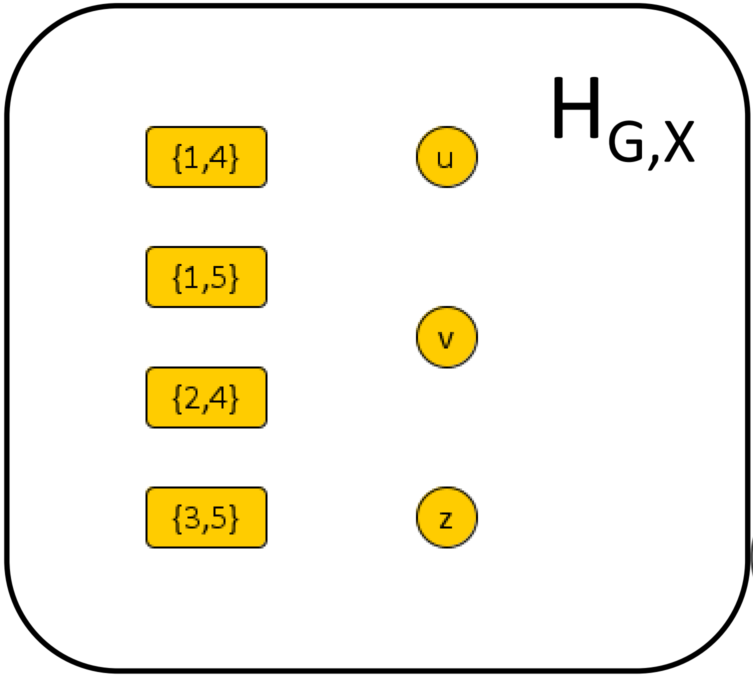

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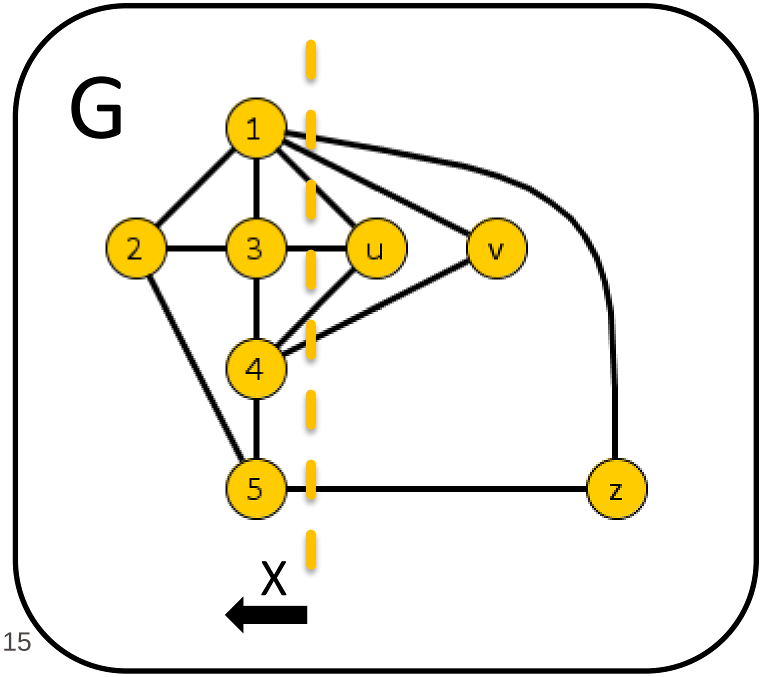

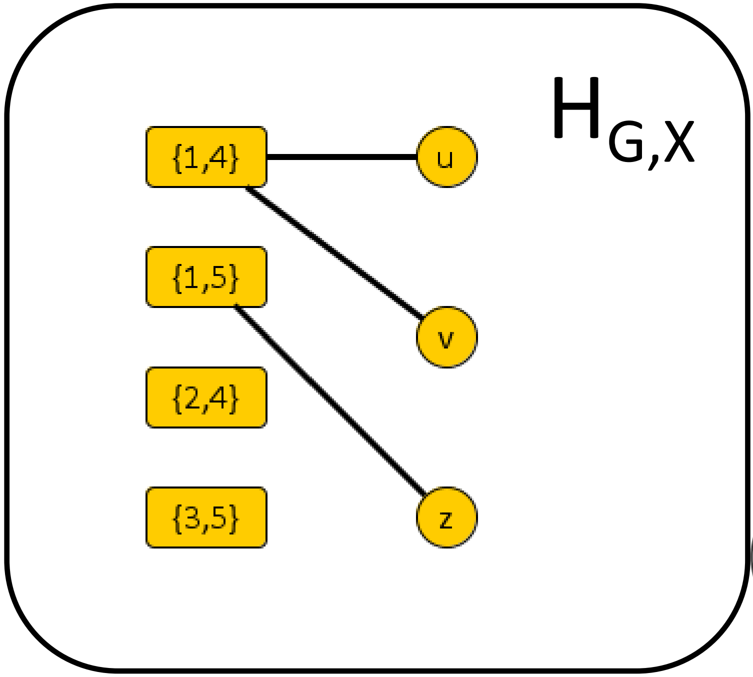

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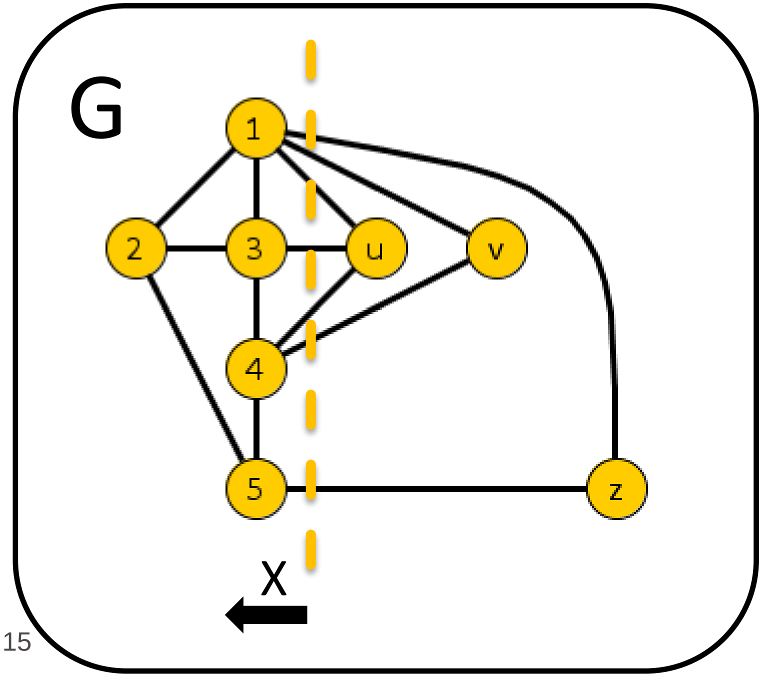

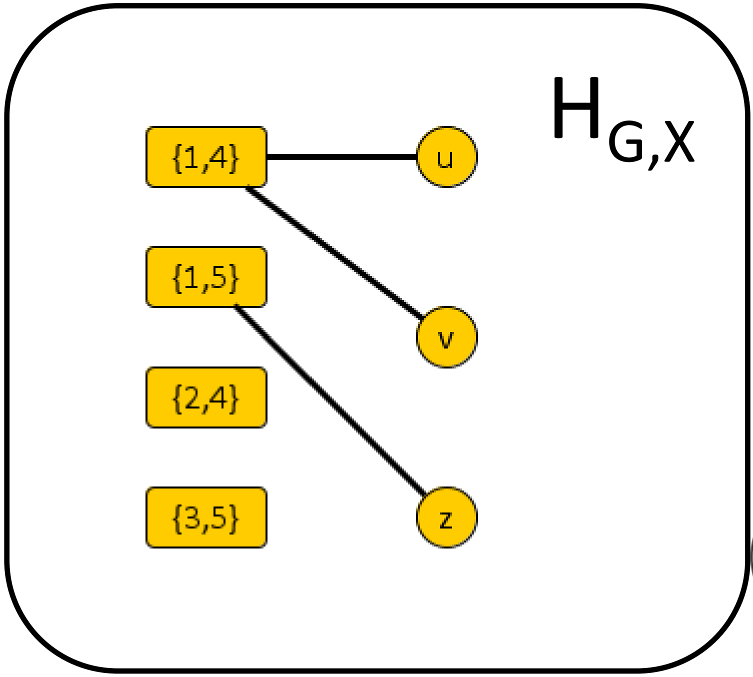

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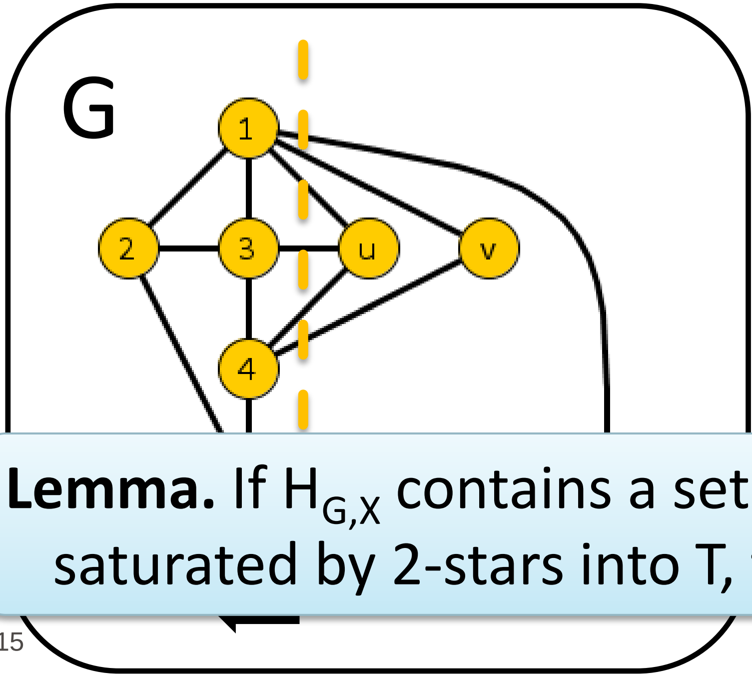

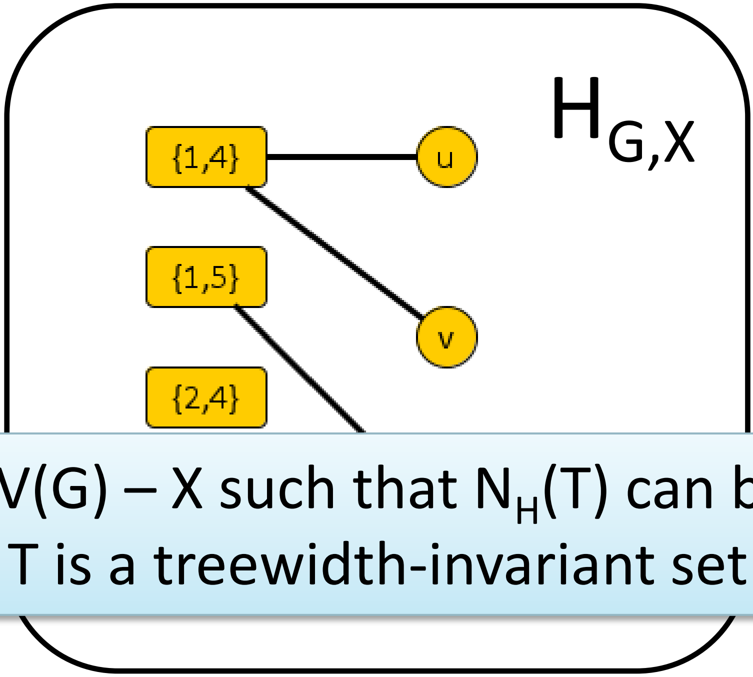

Lemma. If $H_{G,X}$ contains a set T ⊆ V(G) – X such that $N_H(T)$ can be saturated by 2-stars into T, then T is a treewidth-invariant set

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Theorem. TREEWIDTH [VC] has a kernel with $|X|^2$ vertices that can be encoded in $\mathcal{O}(|X|^3)$ bits

Conclusion

- Main contributions

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- Open problems

1. Are there graphs whose edge-count is superquadratic in their vertex cover number, which do not have treewidth-invariant sets?

2. Which problems admit nontrivial polynomial-time sparsification?

3. Does TREEWIDTH [VC] have a kernel of bitsize $\mathcal{O}(|X|^2)$?

4. Does PATHWIDTH [VC] have a kernel with $\mathcal{O}(|X|^2)$ vertices?

Thank you!

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    - So $\mathcal{T}$ has a bag containing $N_G(v)$
    - Append a new bag with $N_G(v) \cup \{v\}$, of size $\leq \Delta(T) + 1$
    - Update independently for each $v \in T$