FPT is Characterized by Useful Obstruction Sets

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Joint work with
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A NEW CHARACTERIZATION OF FPT

Well-Quasi-Orders

- A **quasi-order** is a transitive, reflexive, binary relation $\preceq$ on a (usually infinite) set S.
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- A quasi-order is **polynomial** if $x \preceq y$ can be tested in poly(|x|+|y|) time.

The Obstruction Principle

If $\preceq$ is a WQO on S, and L ⊆ S is a lower ideal, then there is a **finite obstruction set** OBS(L) ⊆ S, such that for all x ∈ S:

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- Any element y ∈ S \ L is an **obstruction.**
- An obstruction is **minimal** if all elements strictly preceding it belong to L.

Algorithmic Applications of WQO's

- Fellows & Langston, JACM 1988:
  - *k*-PATH,
  - *k*-VERTEX COVER,
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- Results led to the development of parameterized complexity.

The class FPT

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- A **parameterized problem** is a set $Q \subseteq \Sigma^* \times \mathbb{N}$
  - Each instance $(x,k) \in \Sigma^* \times \mathbb{N}$ has a *parameter k*.
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Strongly Uniform FPT (Fixed-Parameter Tractable)
A parameterized problem $\mathbb{Q}$ is strongly uniform FPT if there is an algorithm that decides whether $(x,k) \in \mathbb{Q}$ in $f(k)|x|^c$ time.
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- There are weaker notions. (non-uniform, non-computable f)

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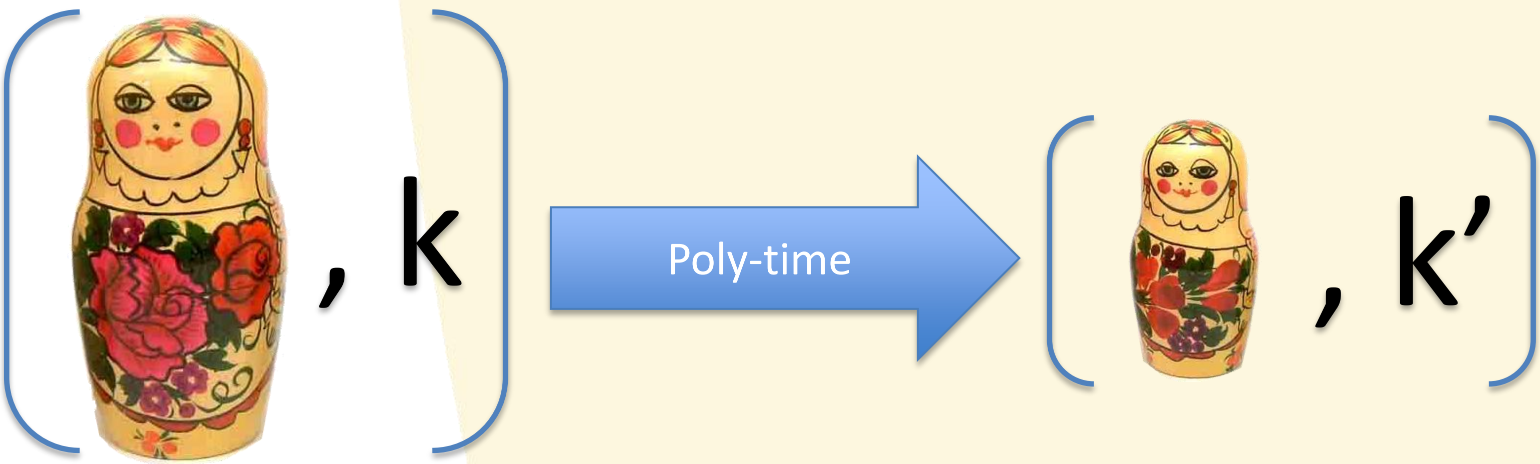

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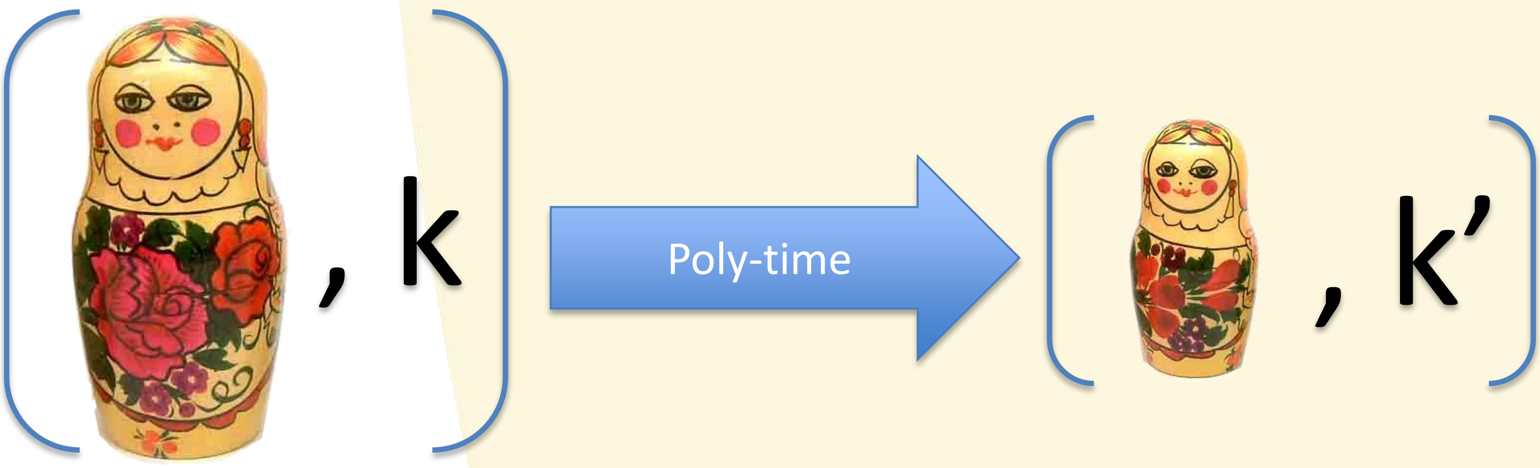

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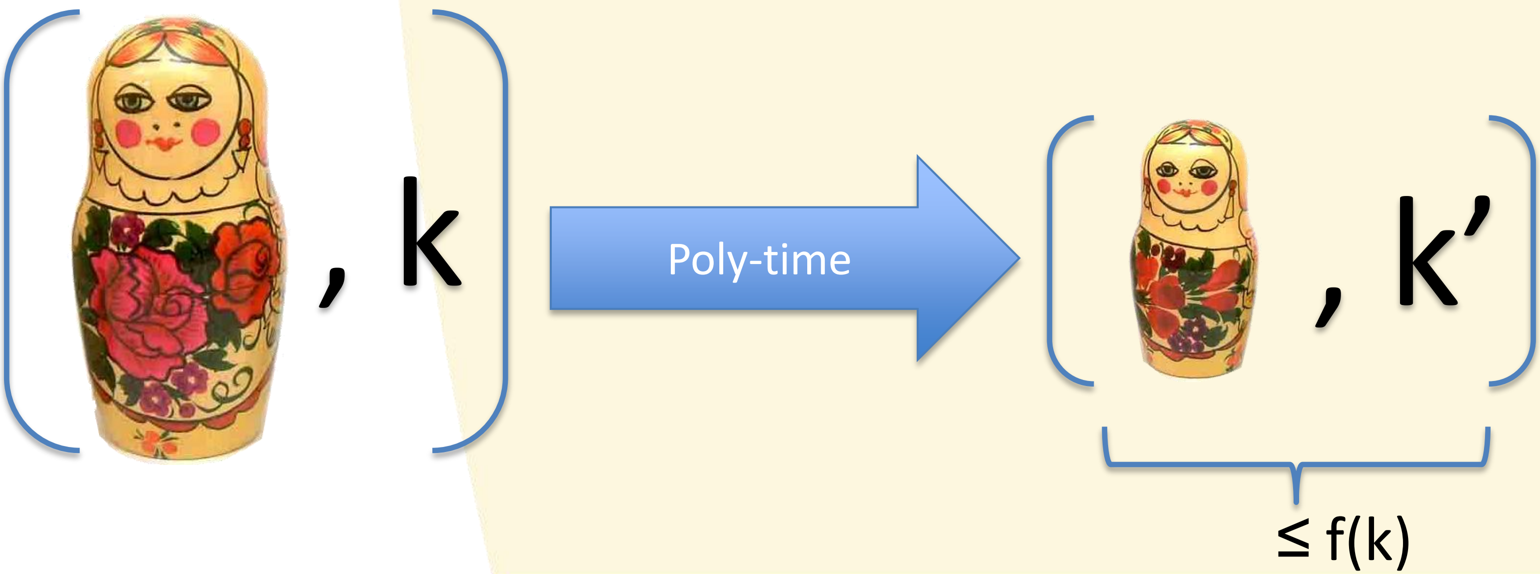
  - and |x’|+k’ is bounded by f(k).



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3. Problem $\mathbb{Q}$ is decidable and there is a polynomial-time **quasi-order** $\preceq$ on $\Sigma^* \times \mathbb{N}$ and a computable function f: $\mathbb{N} \rightarrow \mathbb{N}$ such that:
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- The obstruction-testing method that lies at the origins of FPT is not *just one way* of obtaining FPT algorithms:
  - **all of FPT** can be obtained this way.

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KERNEL SIZE VS. OBSTRUCTION SIZE

Small kernels yield small obstructions

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Reverse is false, assuming NP $\not\subset$ coNP/poly.
(Kratsch & Walhström, 2011)

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Obstruction size vs. kernel size

Polynomial bounds

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k-Vertex Cover

- Best known kernel has 2k – o(k) vertices [Lampis'11]
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q-Coloring parameterized by Vertex Cover

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- Vertex-minimal no-instances have $\text{vc}^{\Theta(q)}$ vertices.

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EXPLOITING OBSTRUCTIONS FOR LOWER-BOUNDS ON KERNEL SIZES

Composition algorithms

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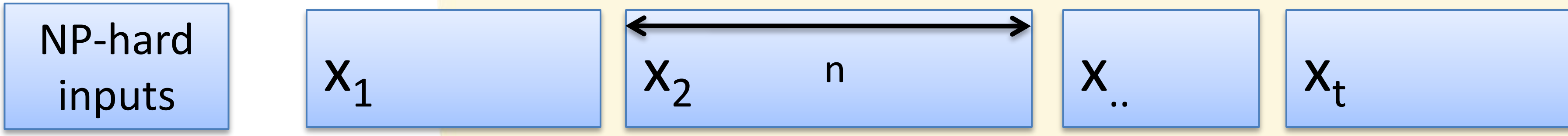
NP-hard inputs

x_1 | x_2 | $x_{..}$ | x_t

Composition algorithms



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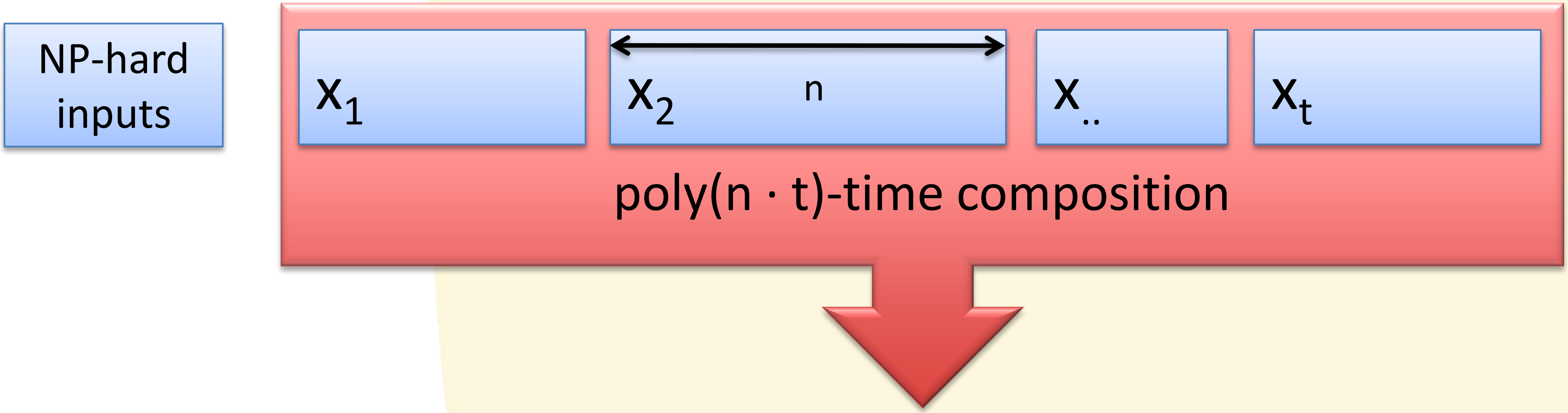

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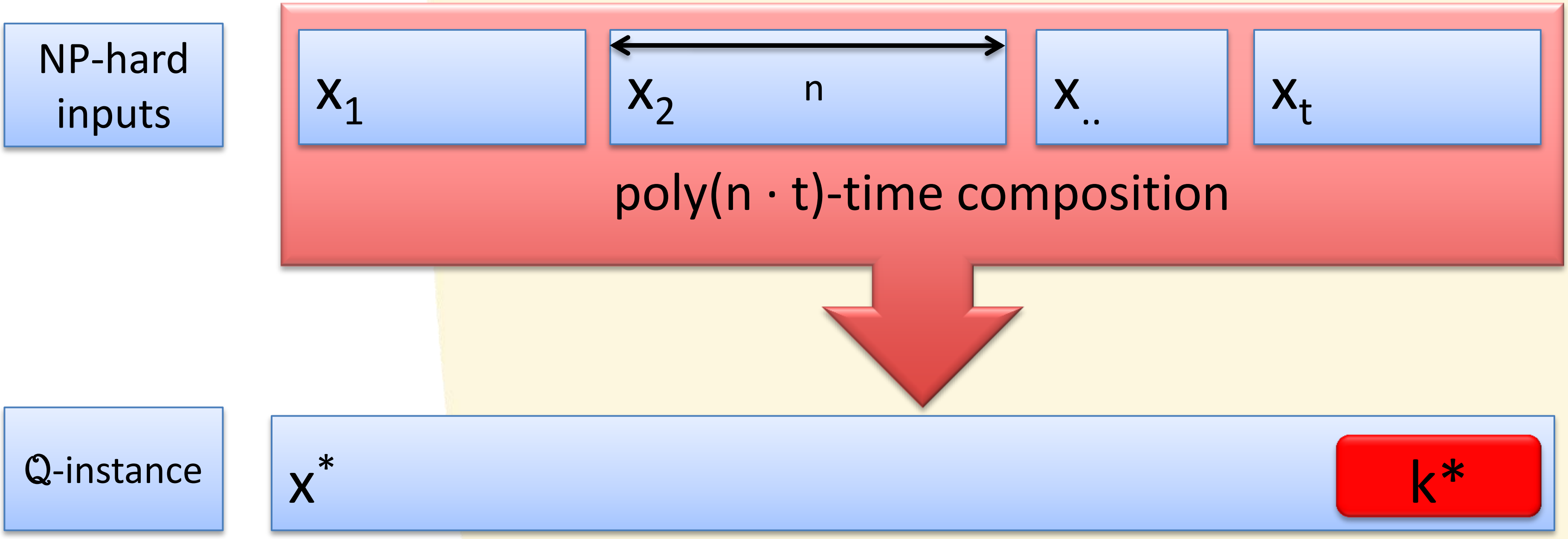

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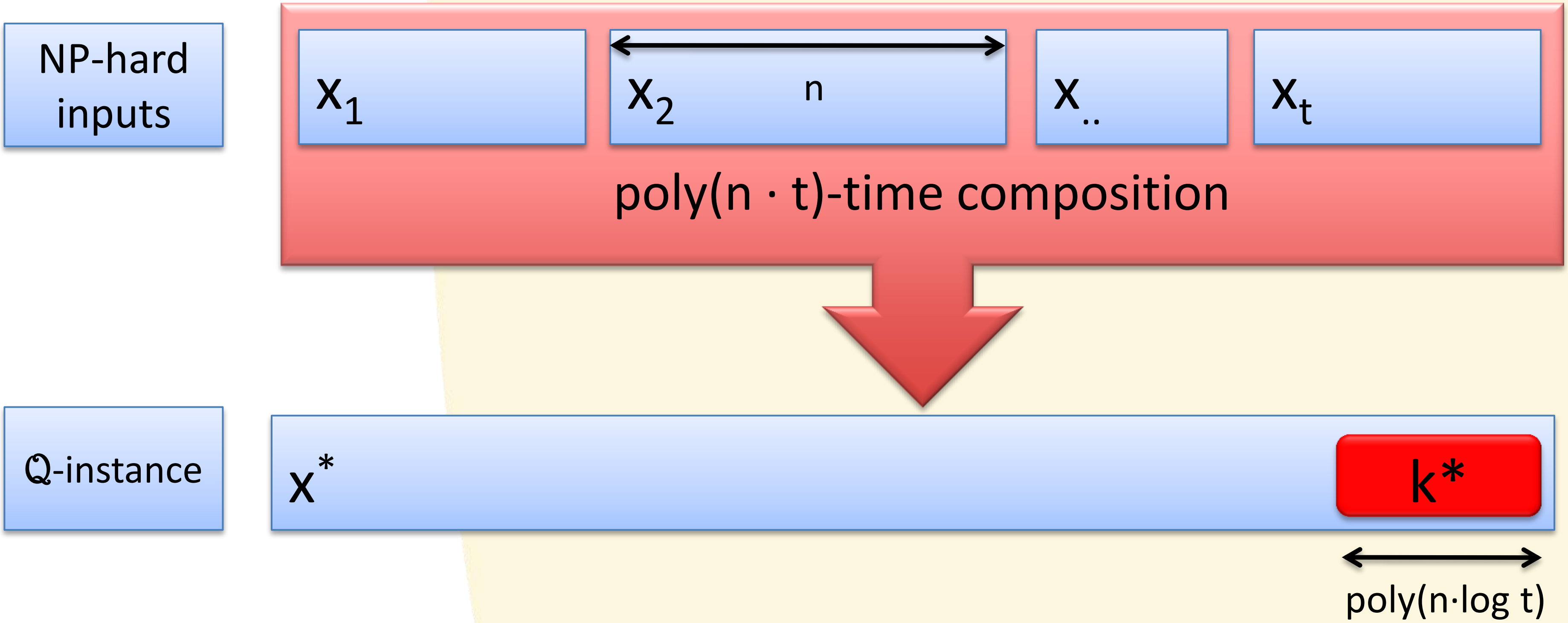

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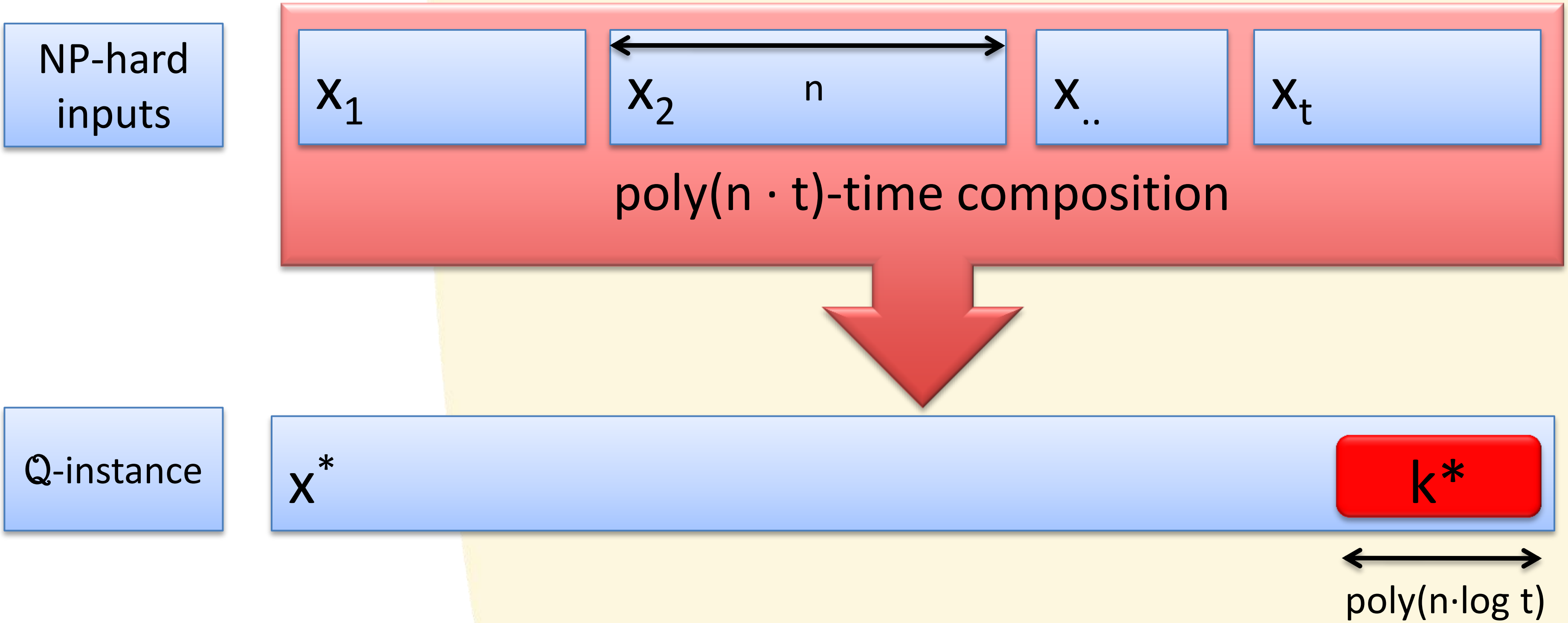

AND-Cross-composition: (x*,k*) ∈ Q iff all inputs are YES

OR-Cross-composition: (x*,k*) ∈ Q iff some input is YES

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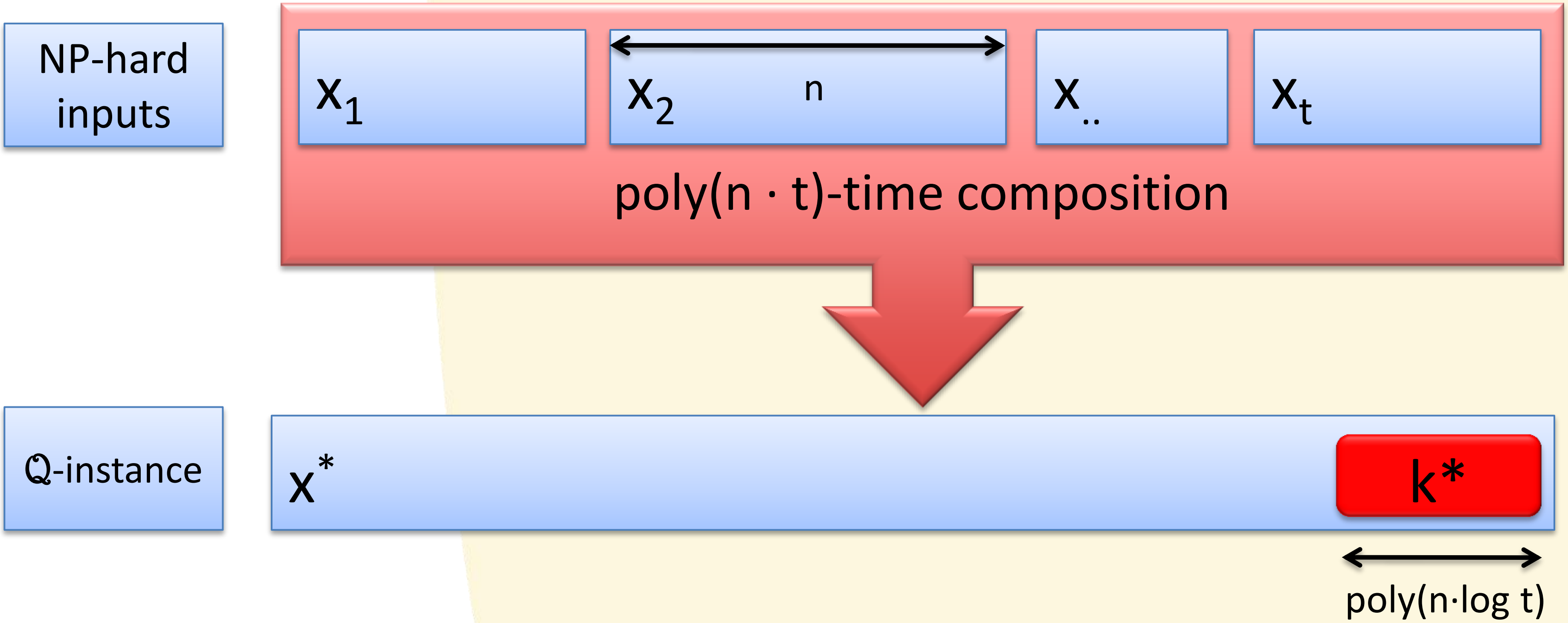

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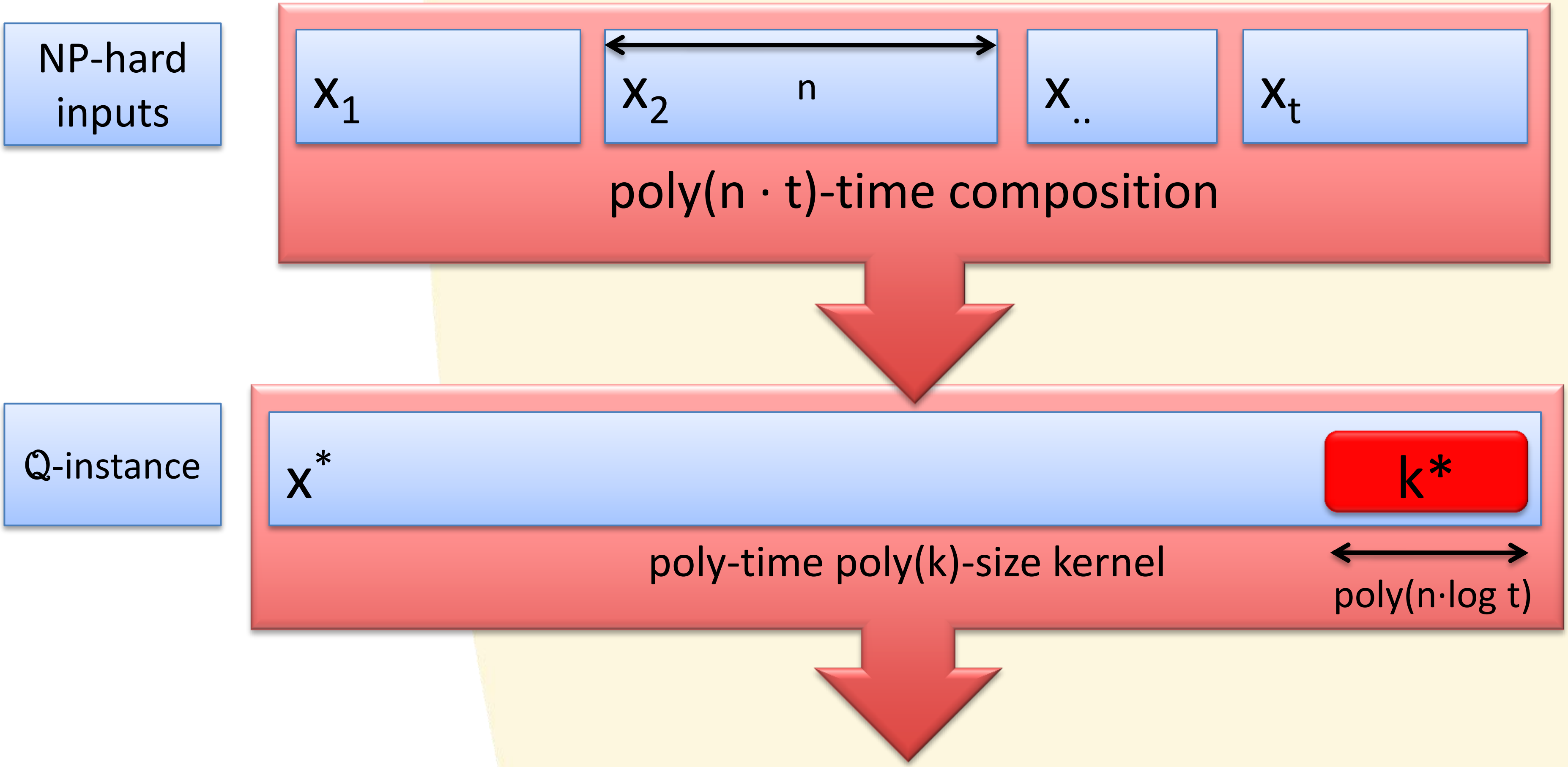

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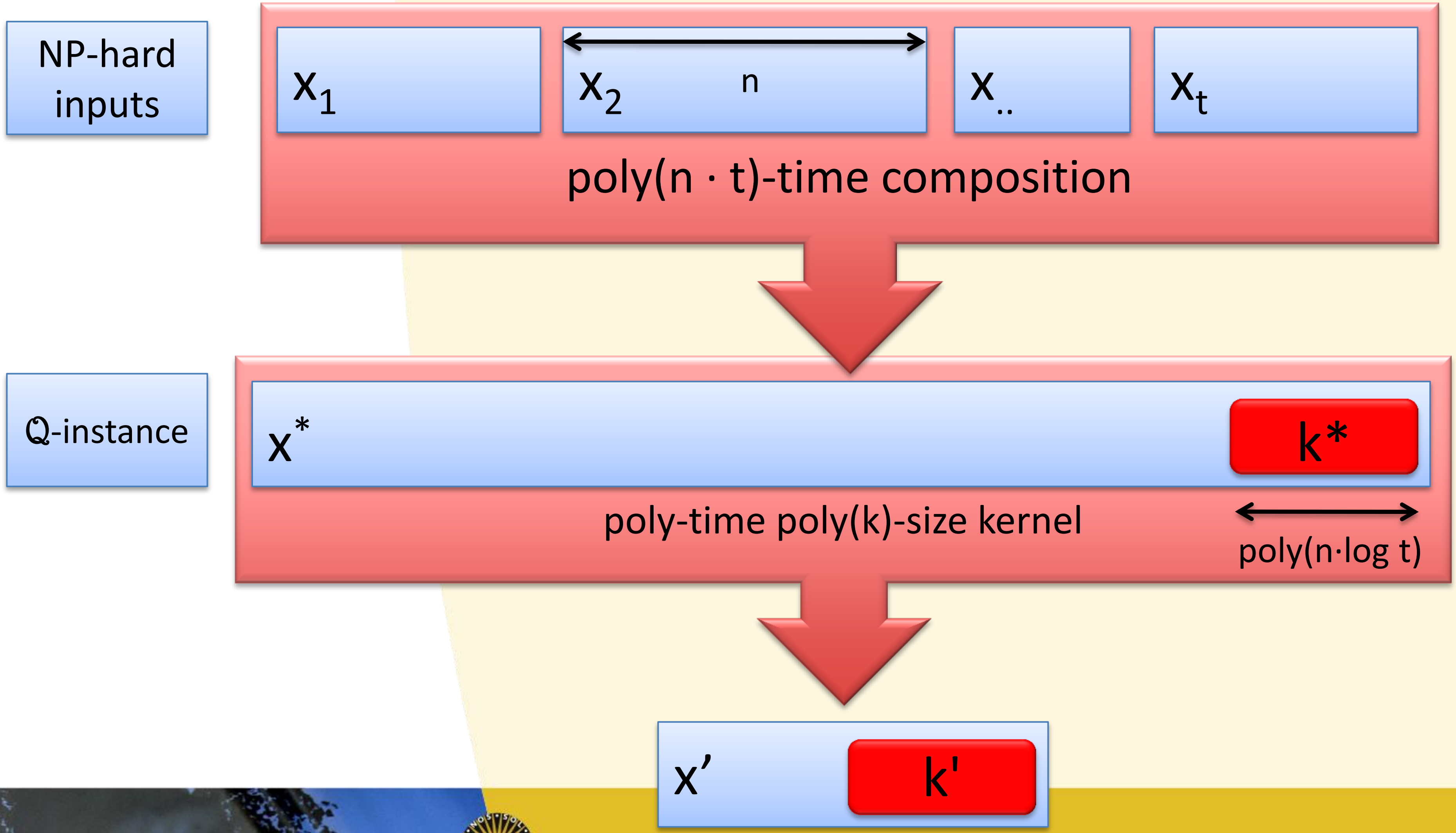

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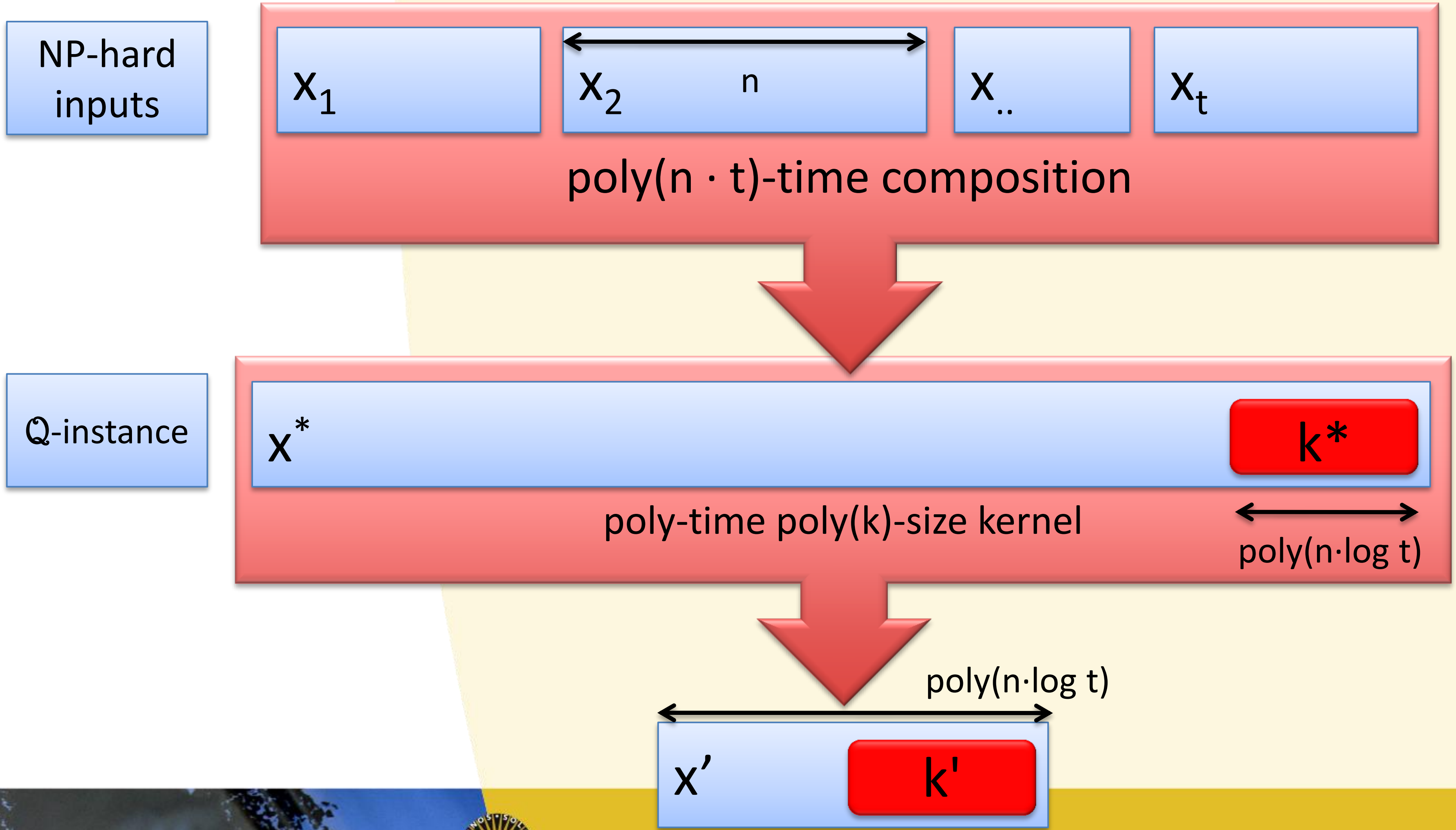

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k-PATHWIDTH is AND-compositional and does not admit a polynomial kernel unless NP ⊆ coNP/poly. [BodlaenderDFH’09,Drucker’12]

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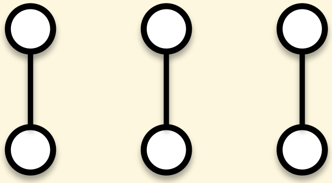

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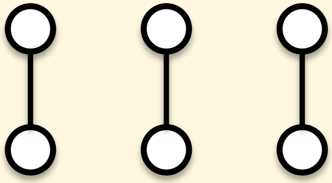
Joining 3 minimal tree-obstructions to PW=k, gives minimal obstruction to PW=k+1



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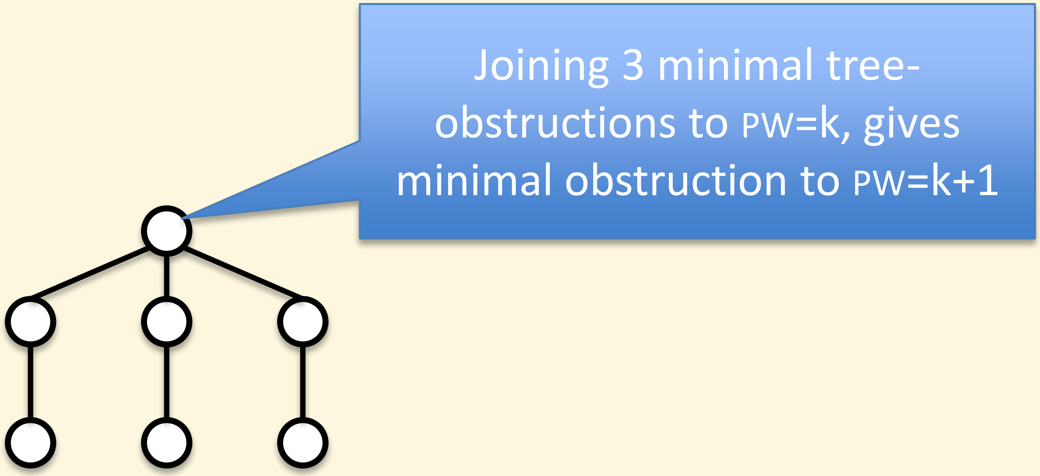

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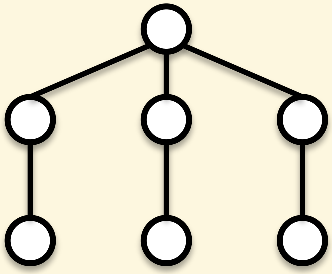

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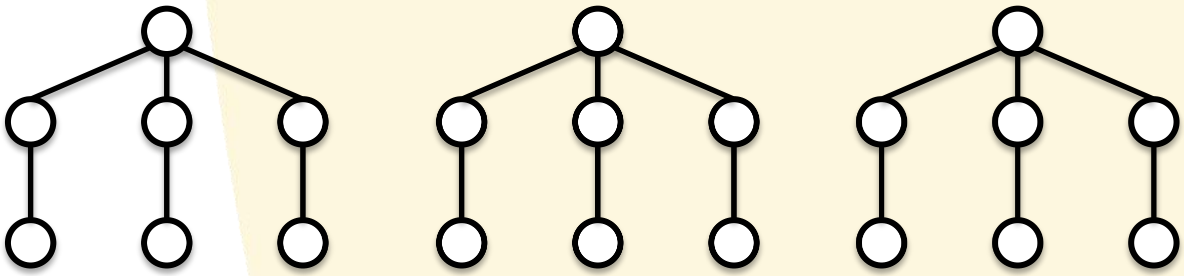

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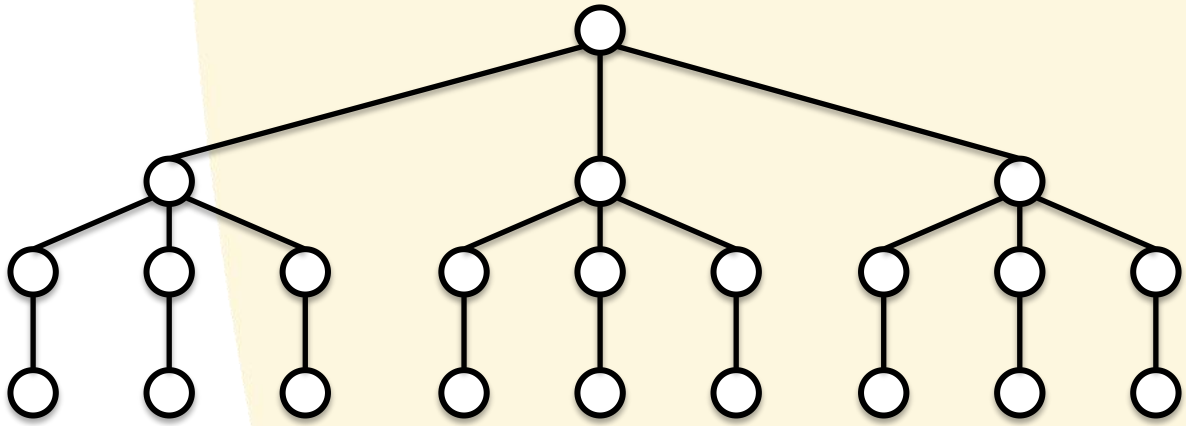

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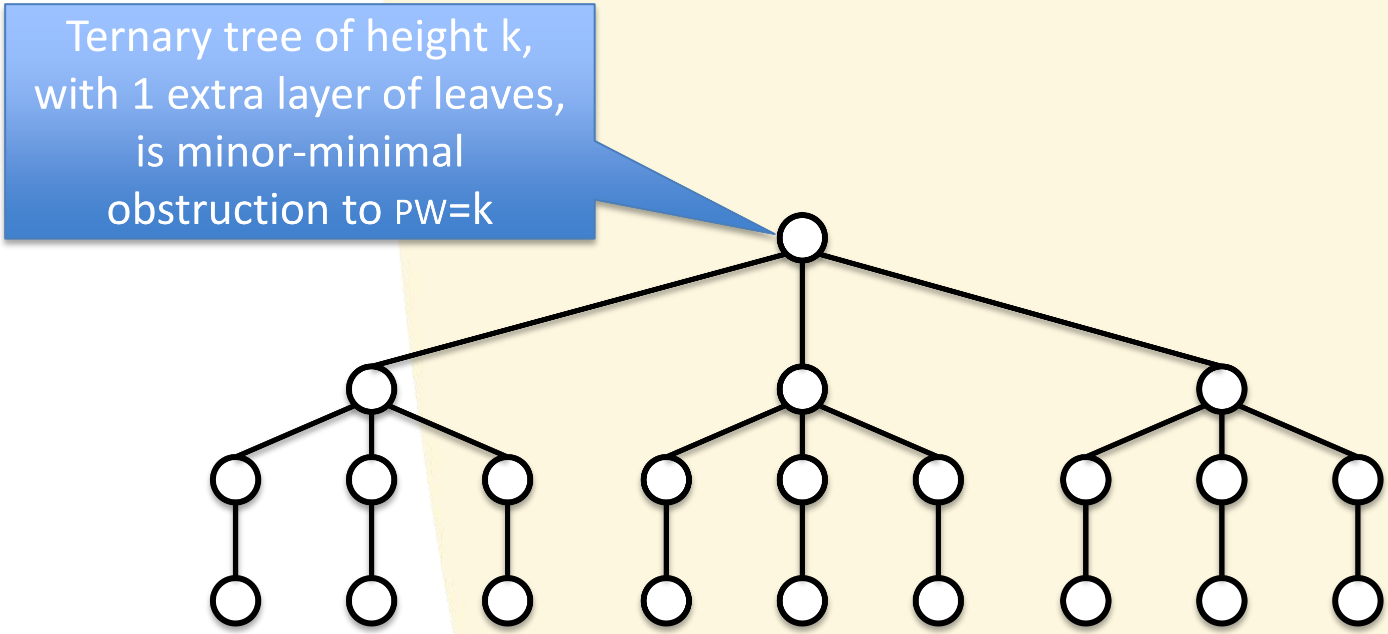

Construction

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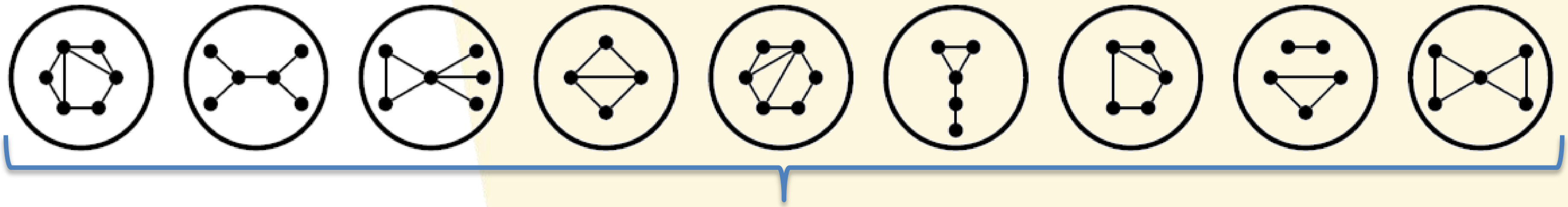
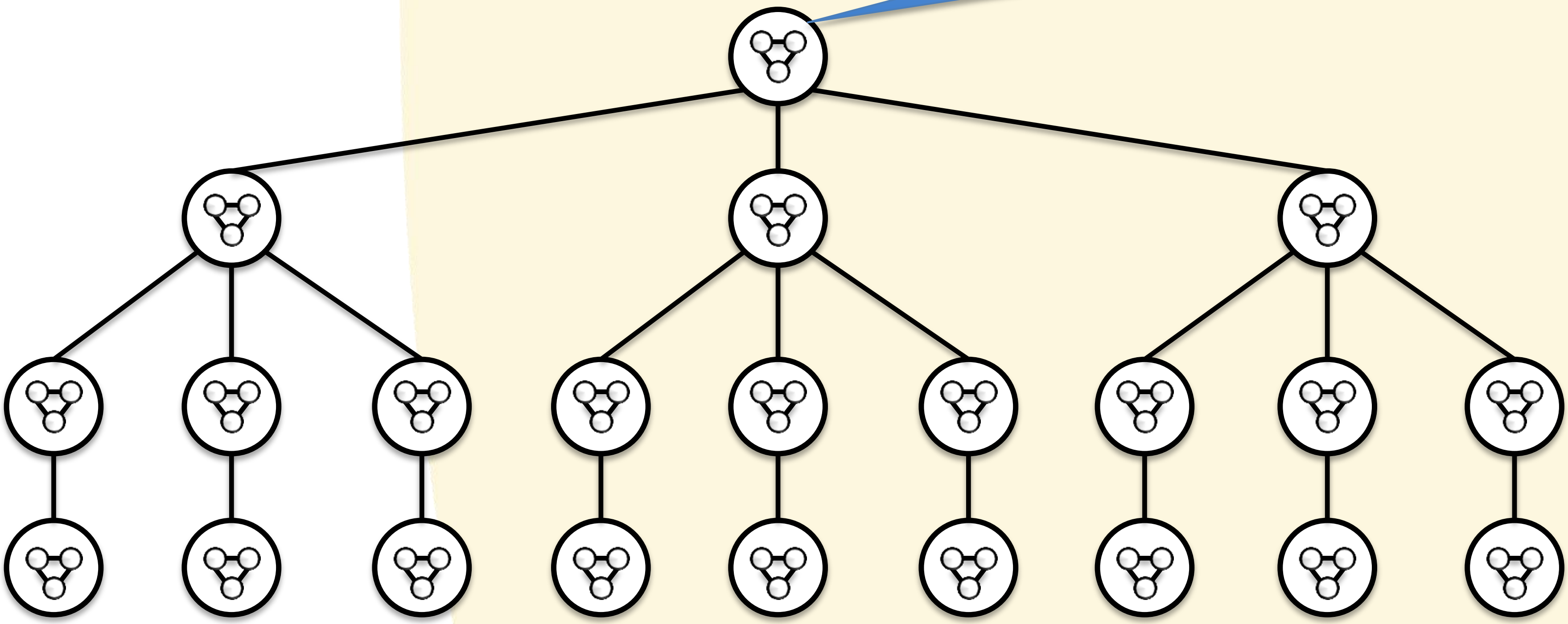


$t=3^s$ instances of PW-IMPROVEMENT with $k=3$
(each asking if $\text{PW}(G_i) \leq k - 2$)

Construction

Obstruction with 3^s leaves,
inflated by factor k
Pathwidth is $\mathcal{O}(k \cdot s) \leq \mathcal{O}(n \cdot \log t)$



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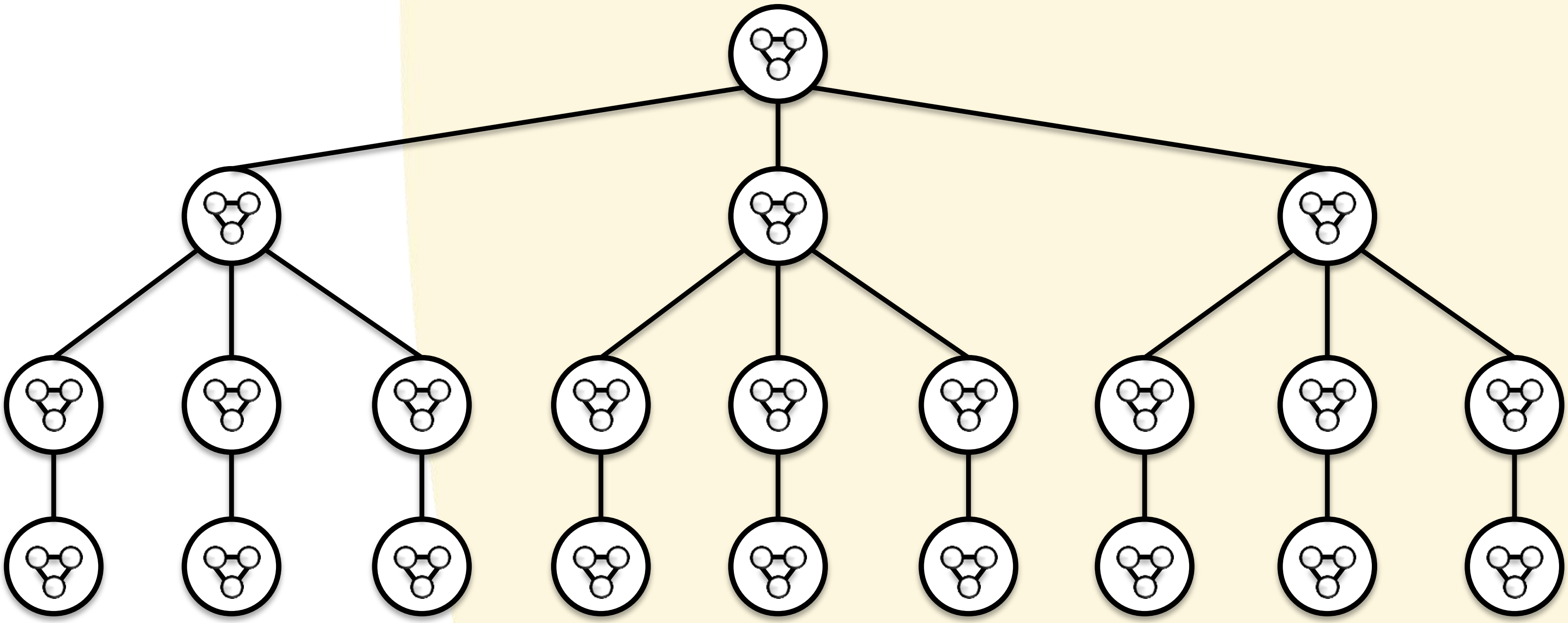

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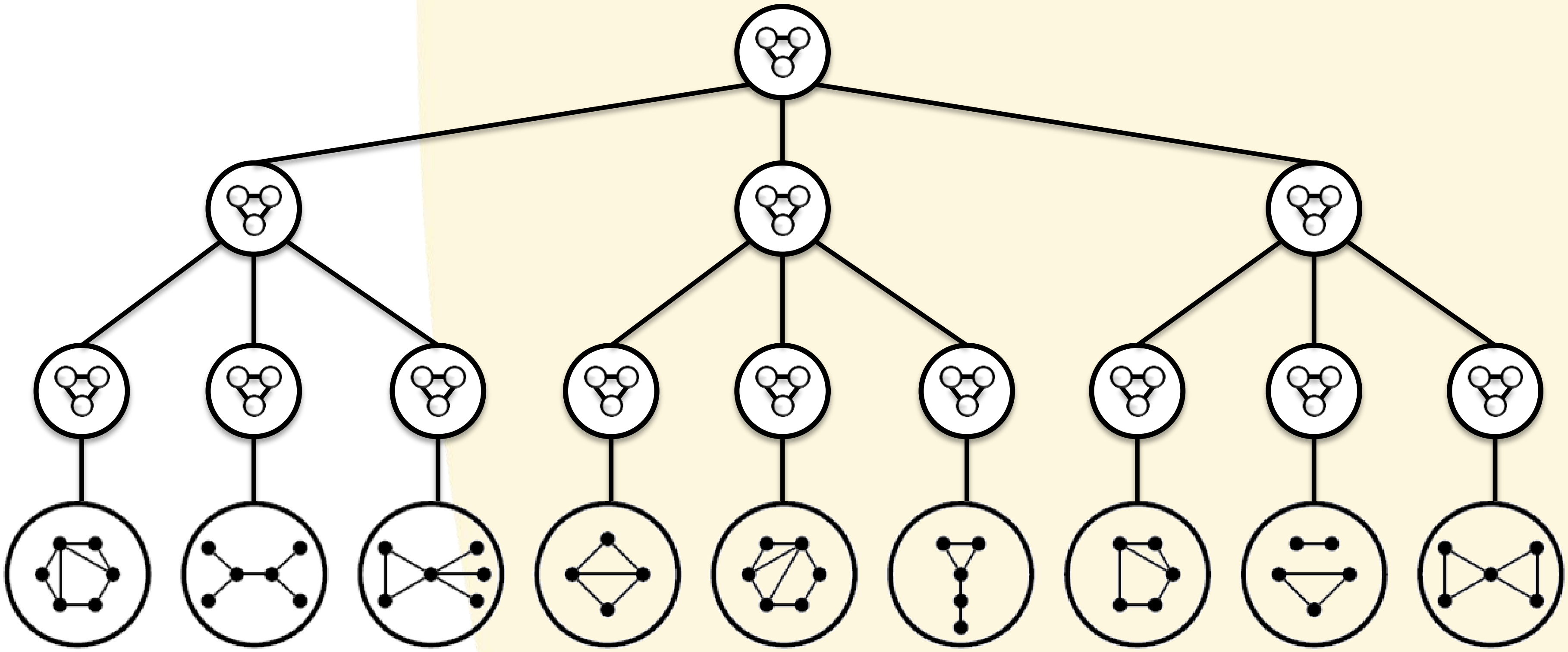
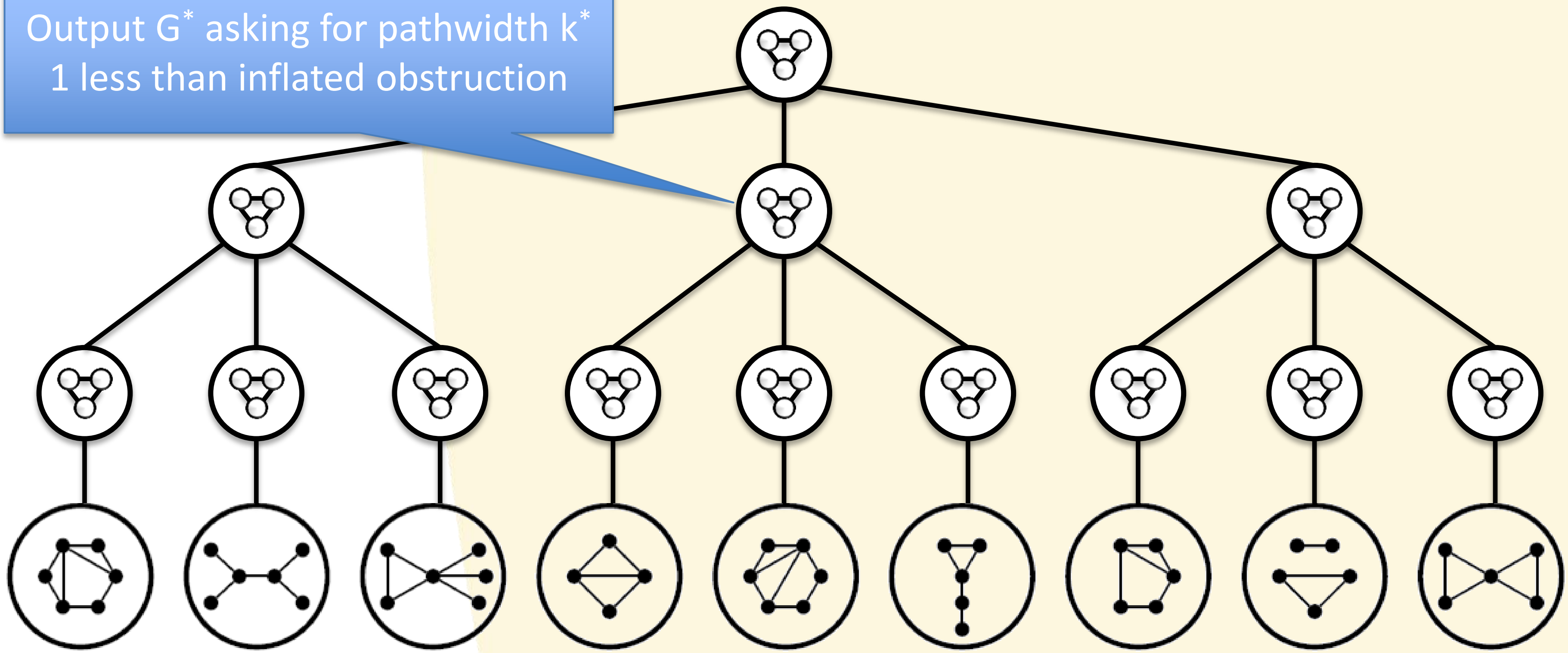
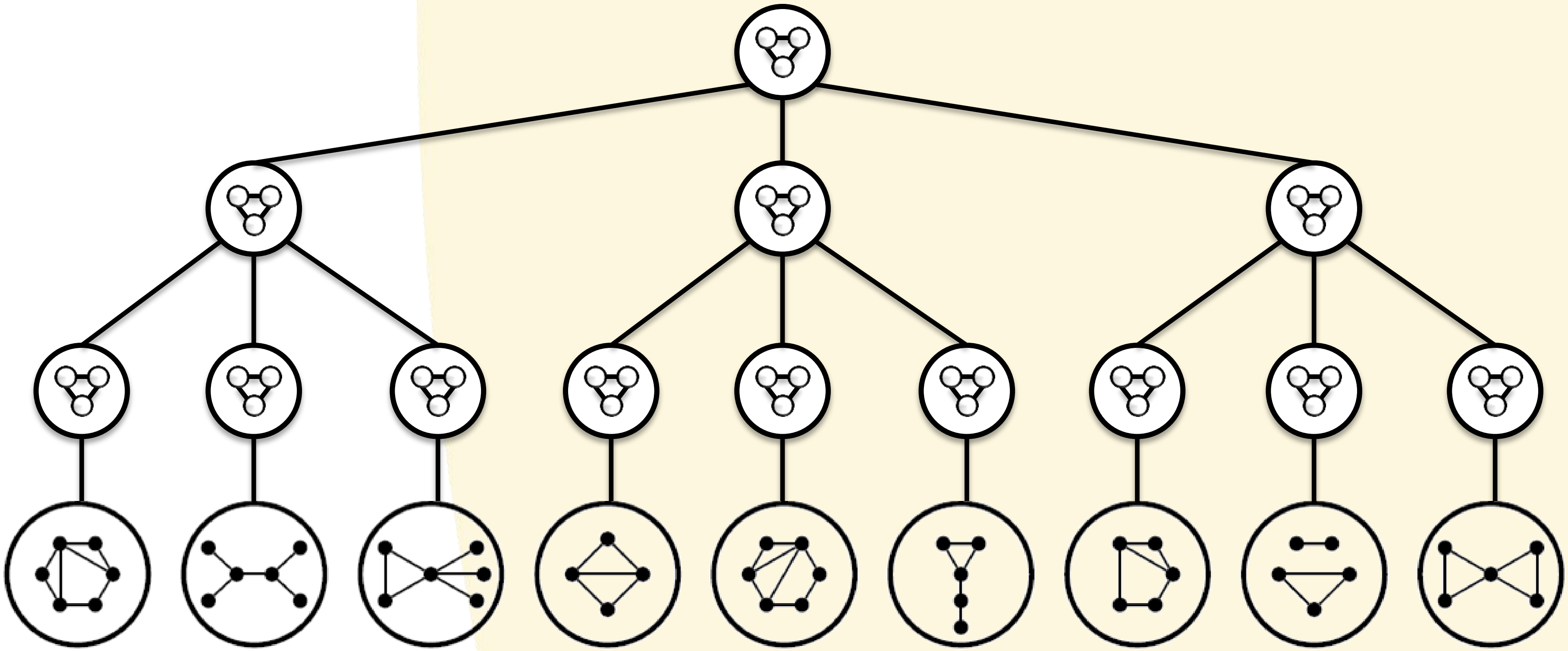


Construction

Output G^* asking for pathwidth k^*
1 less than inflated obstruction



Correctness sketch



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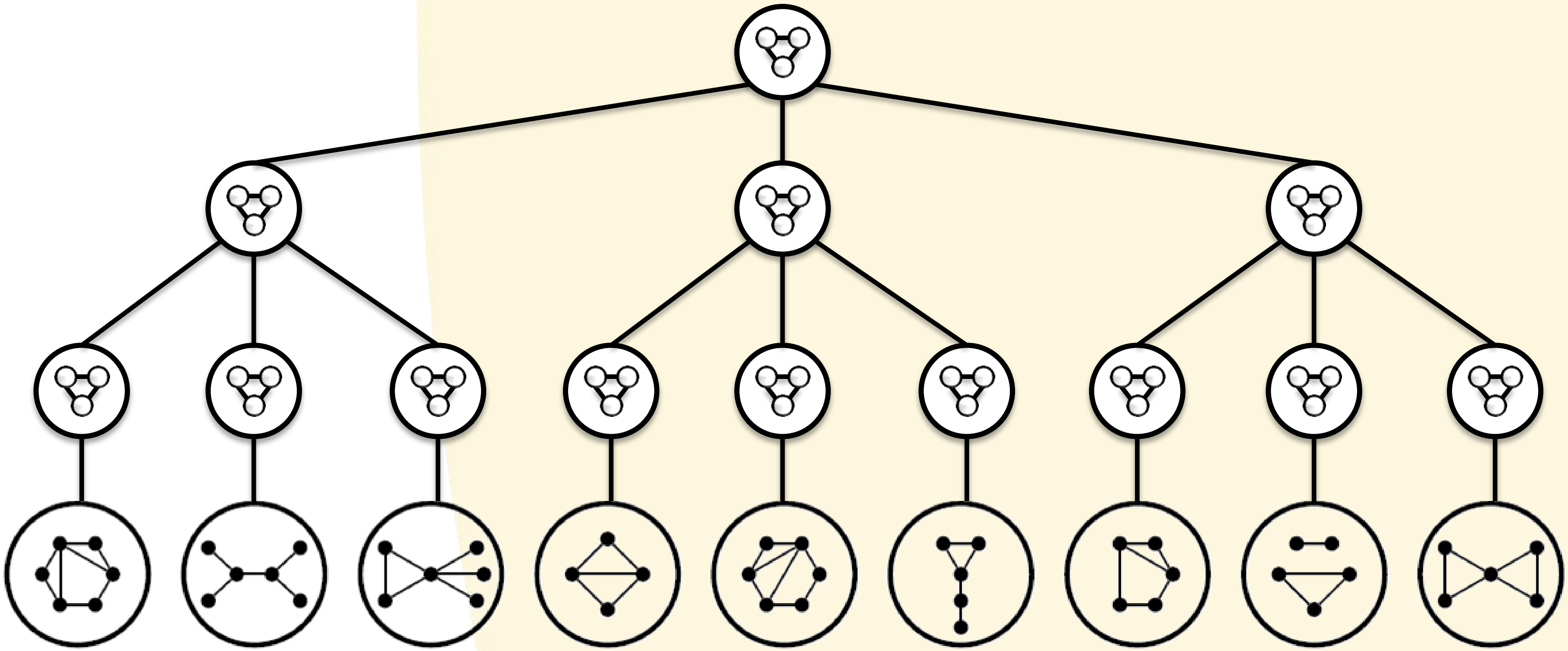

Claim: **some** input i has $\mathrm{PW}(G_i) \leq k\text{-}2 \rightarrow \mathrm{PW}(G^*) < \mathrm{PW}(T^s \Diamond k)$

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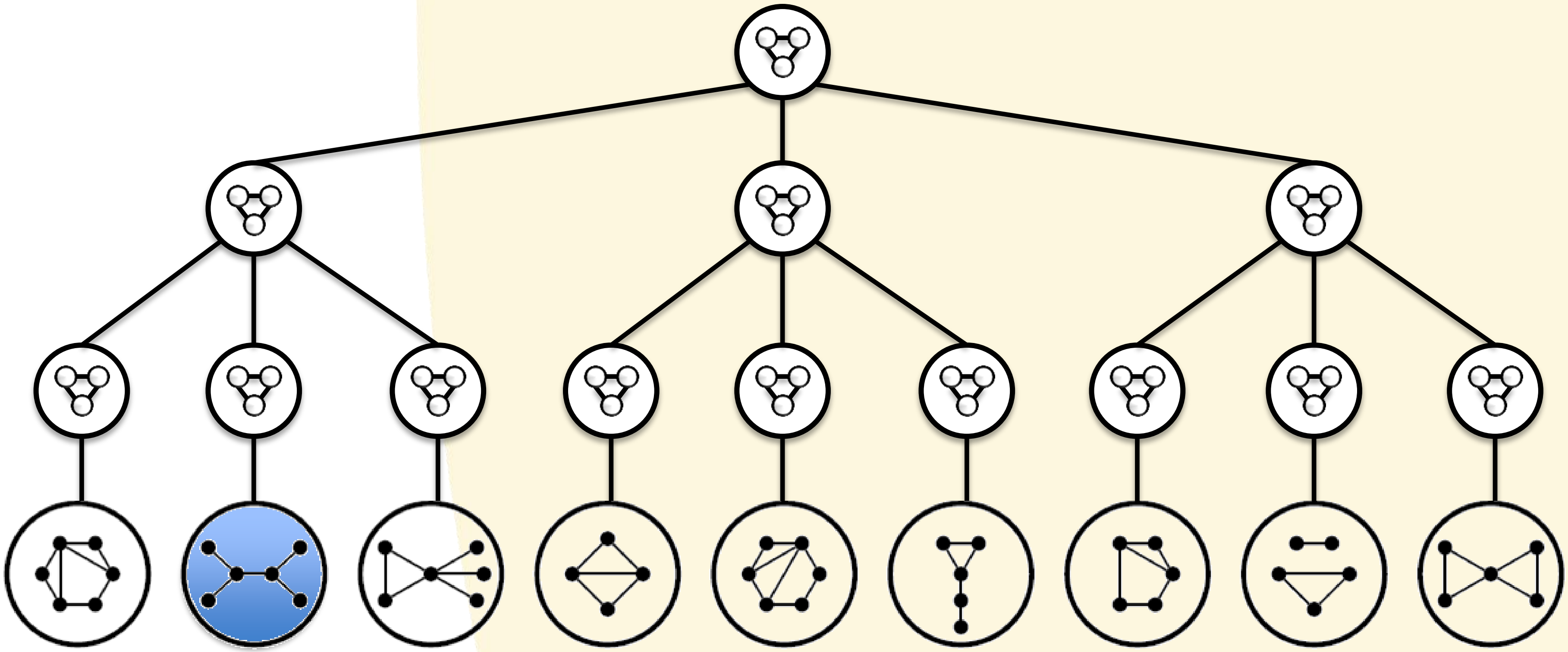
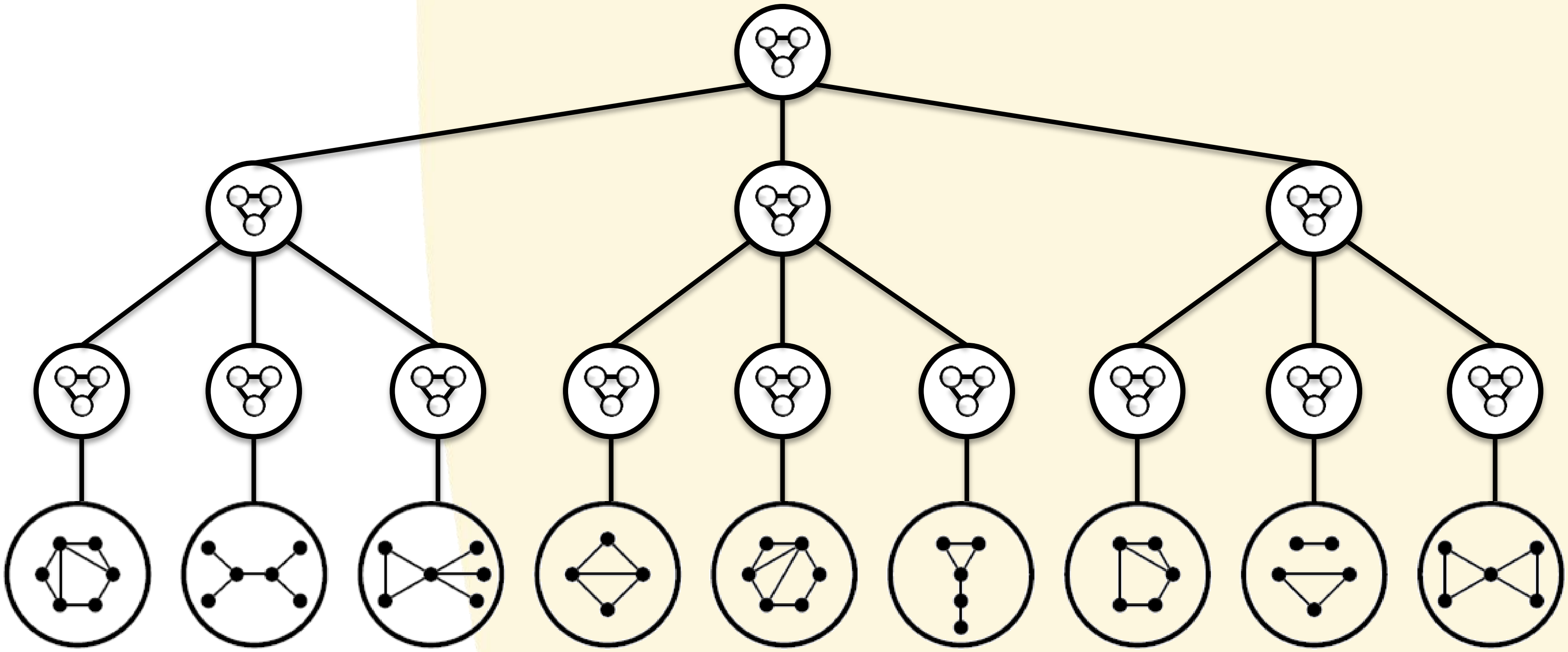


Claim: **some** input i has PW(G_i) ≤ k-2 → PW(G^*) < PW(T^s ◊ k)

Correctness sketch



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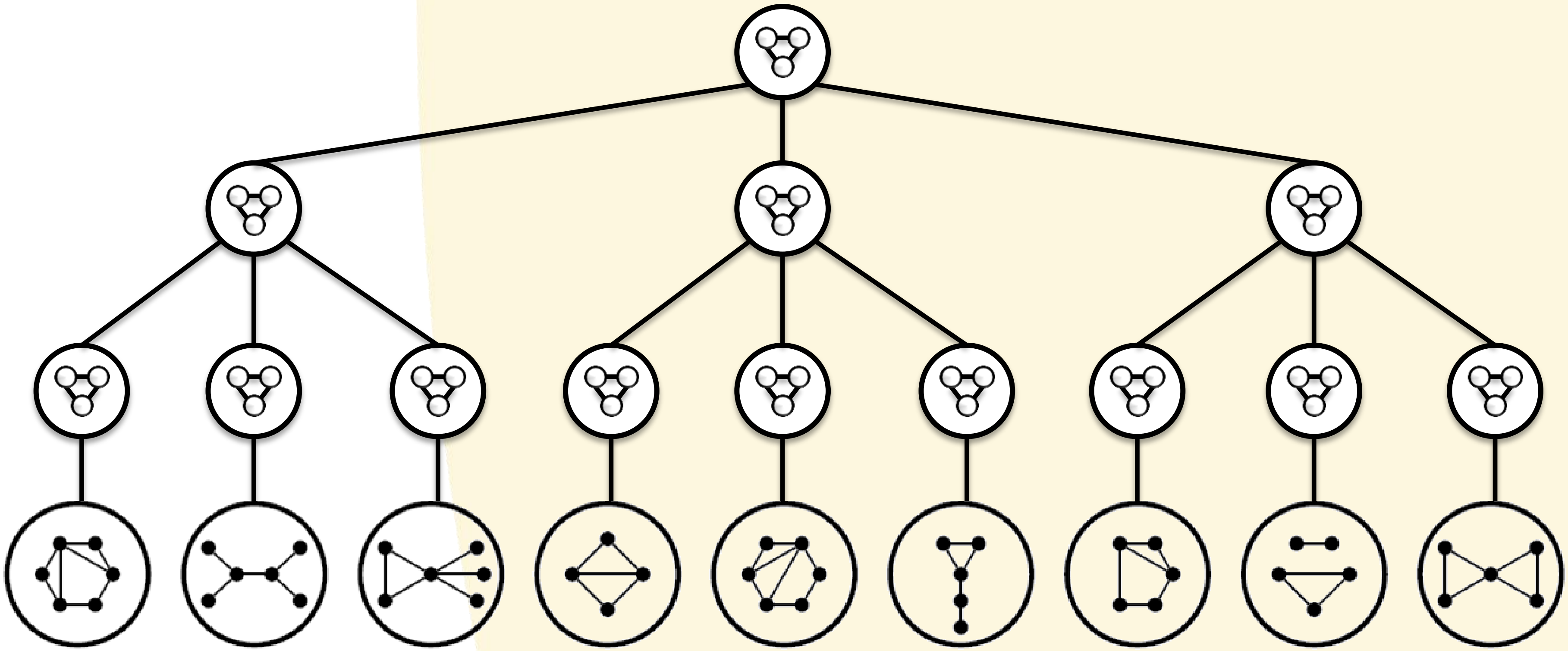

Claim: **all** inputs have PW(G_i) > k-2 ➔ PW(G^*) ≥ PW(T^s ◊ k)

Conclusion

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OR-Cross-composition into *k*-TREEWIDTH?

Further relations between kernel and obstruction sizes?

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THANK YOU!

- Open prob…

OR-Cross-composition into *k*-TREEWIDTH?

Further relations between kernel and obstruction sizes?